Modelling Multivariate Extreme Dependence
with Application to Financial Contagion

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Motivation

**Theoretical:**
- Multivariate Extreme Value theory
  - Pros: $d \geq 2$; various parametric models
  - Cons: limited dependence structures
- Ledford-Tawn approach
  - Pros: rich dependence structures
  - Cons: only for $d=2$; sparse parametric models (only one proposed by Ramos and Ledford (2008))

**Practical:** Global financial crises triggered by the US’ subprime mortgage crisis

**Goal:**
- General approach to derive multivariate parametric models under Ledford-Tawn framework; examples
- Financial application
Multivariate Extreme Value model

Define i.i.d $p$-dimensional random vectors: $\mathbf{X}_k = (X_{1,k}, \cdots, X_{p,k})'$, $k = 1, \cdots, n$ with marginal distribution $\Pr(X_{i,k} \leq x_i) = F_{X_i}(x_i)$, $i = 1, \cdots, p$.

The $i$th component maximum: $M_{i,n} = \max(X_{i,1}, \cdots, X_{i,n})$

Choose normalising constants $\mathbf{a}_n = (a_{1,n}, \cdots, a_{p,n})'$ and $\mathbf{b}_n = (b_{1,n}, \cdots, b_{p,n})'$ such that,

$$\lim_{n \to \infty} \Pr \left\{ \frac{M_{i,n} - b_{i,n}}{a_{i,n}} \leq x_i, i = 1, \cdots, p \right\} = G(x_1, \cdots, x_p)$$

Denoting by vectors, we get:

$$\lim_{n \to \infty} \Pr \left\{ \frac{M_n - \mathbf{b}_n}{\mathbf{a}_n} \leq \mathbf{x} \right\} = G(\mathbf{x})$$

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Multivariate Extreme Value model

It is easily checked that all the marginal distributions of $G$ are Generalised Extreme Value distribution (GEV), which belongs to one of the following three families (Fisher-Tippett theorem).

\[
F_{X_i}(x_i) = G(\infty, \cdots, \infty, x_i, \infty, \cdots, \infty) = \begin{cases} 
\text{Gumbel} : & \exp \{- \exp(-x_i)\} \\
\text{Fréchet} : & \exp \{-x_i^{-\alpha}\} \\
\text{Weibull} : & \exp \{-(-x_i)^\alpha\}
\end{cases}
\]

A common practice is to choose the Fréchet marginals, and usually the unit Fréchet case when shape parameter $= 1/\alpha = 1$.

This is WOLG, as any marginal distributions can be changed into a unit Fréchet by a probability integral transformation. Details can be found in Smith (2003).
Motivation

Multivariate Extreme Value model

Pickands’ representation: For $\mathbf{x}$ with unit Fréchet marginals, $\mathbf{x}$ follows a multivariate extreme value distribution iff its joint distribution function can be represented as:

$$G(\mathbf{x}) = \exp \{-V(\mathbf{x})\}$$ (1)

Here, $V(\mathbf{x})$ is the exponent measure as in de Haan and Resnick (1977) and can be expressed in terms of an angular measure $H$ (density $h$) as:

$$V(\mathbf{x}) = \int_{S_p} \max_{1 \leq i \leq p} \left( \frac{W_i}{X_i} \right) dH(\mathbf{w})$$ (2)

Qin, Smith, Ren
Multivariate Extreme Value model

Where, $H$ is a positive finite measure on the $(p - 1)$-simplex,

$$S_p = \left\{ (w_1, \ldots, w_p) : \sum_{j=1}^{p} w_j = 1, w_j \geq 0, j = 1, \ldots, p \right\}$$

satisfying the following constraint.

**MEV constraint**: Suppose $x_i = u, x_j = \infty, j \neq i, i = 1, \ldots, p$, then,$V(\infty, \cdots, \infty, u, \infty, \cdots, \infty) = u^{-1}$. Thus,

$$1 = \int_{S_p} w_i \, dH(w) = V(\infty, \cdots, \infty, 1, \infty, \cdots, \infty)$$  \hfill (3)

wherever the 1 is.
Limitation of the Multivariate Extreme Value model

As stated in Ledford and Tawn (1996), all Bivariate Extreme Value models with Fréchet marginals have:

$$\lim_{z \to \infty} \Pr(Z_1 > z, Z_2 > z) \propto \begin{cases} z^{-1} & \text{asymptotic dependence} \\ z^{-2} & \text{exact independence} \end{cases}$$

However, strictly speaking, there should also be negative association and positive association under asymptotic independence.
Ledford-Tawn model 96

In Ledford and Tawn (1996), for \((Z_{1k}, Z_{2k}), k = 1, \ldots, n\) with unit Fréchet marginals but unknown dependence structure, the joint survival function \(\bar{F}_{Z_1Z_2}\) has the following asymptotic form:

\[
\bar{F}_{Z_1Z_2}(z, z) = L(z)z^{-1/\eta}
\]

where, \(L(z)\) is a slowly varying function, i.e., \(\lim_{z \to \infty} \frac{L(tz)}{L(z)} = 1, \forall t > 0\), \(\eta \in (0, 1]\) is called the **Coefficient of tail dependence**.

\[
\begin{align*}
\eta = 1, & \quad \text{asymptotic dependence} \\
\frac{1}{2} < \eta < 1, & \quad \text{positive association} \\
\frac{1}{2}, & \quad \text{exact independence} \\
0 < \eta < \frac{1}{2}, & \quad \text{negative association}
\end{align*}
\]
In Ledford and Tawn (1997), the joint survival function has the asymptotic form as:

\[ \bar{F}_{Z_1, Z_2}(z_1, z_2) = L(z_1, z_2) z_1^{-c_1} z_2^{-c_2} \]

where, \( c_1 + c_2 = \frac{1}{\eta} \), \( L(z_1, z_2) \) is a bivariate slowly varying function (BSV) with limiting function \( g \), i.e., \( \lim_{t \to \infty} \frac{L(tz_1, tz_2)}{L(t, t)} = g(z_1, z_2) \) and \( g(cz_1, cz_2) = g(z_1, z_2), \forall c > 0, z_1, z_2 > 0. \)
Ramos and Ledford’s extension

Ramos and Ledford (2008) reformulated the Ledford-Tawn model:

\[
\tilde{F}_{Z_1Z_2}(z_1, z_2) = \tilde{L}(z_1, z_2) (z_1 z_2)^{-\frac{1}{2\eta}}
\]

where,

\[
\tilde{L}(z_1, z_2) = L(z_1, z_2) \left( \frac{Z_1}{Z_2} \right)^{\kappa/2}, \quad \kappa = c_2 - c_1
\]

Consider the limiting joint survival distribution of the conditional variables \((X_1, X_2) = (Z_1/u, Z_2/u)|(Z_1 > u, Z_2 > u)\), thus, \(X_1, X_2 \geq 1\). Then,

\[
\tilde{F}_{X_1X_2}(x_1, x_2) = \lim_{u \to \infty} \frac{\Pr(Z_1 > ux_1, Z_2 > ux_2)}{\Pr(Z_1 > u, Z_2 > u)}
\]

\[
= \lim_{u \to \infty} \frac{\tilde{L}(ux_1, ux_2)}{\tilde{L}(u,u)} \frac{1}{(x_1 x_2)^{1/2\eta}}
\]

\[
= \frac{g(x_1, x_2)}{(x_1 x_2)^{1/2\eta}} = \frac{g^*(\frac{x_1}{x_1 + x_2})}{(x_1 x_2)^{1/2\eta}}
\]
Ramos and Ledford’s extension

**Ramos-Ledford spectral model:** By taking a polar-coordinate transformation, the following relationship of the limiting survivor in terms of an angular measure $H^*_\eta$ (density $h^*_\eta$) is obtained:

$$\bar{F}_{X_1X_2}(x_1, x_2) = \int_0^1 \eta \left\{ \min\left( \frac{w}{x_1}, \frac{1 - w}{x_2} \right) \right\} \frac{1}{\eta} dH^*_\eta(w)$$

**Ramos-Ledford Constraint:** Suppose $x_1 = x_1 = u$, then,

$$\bar{F}_{X_1X_2}(u, u) = u^{-\frac{1}{\eta}}.$$ The following condition should hold:

$$1 = \int_0^1 \eta \left\{ \min(w, 1 - w) \right\} \frac{1}{\eta} dH^*_\eta(w) = \bar{F}_{X_1X_2}(1, 1)$$

A parametric family based on the asymmetric logistic model is given in Ramos and Ledford (2008), which however seems to be the only one available.
Motivation

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Practical: Global financial crises triggered by the US’ subprime mortgage crisis

Goal:
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- financial application
A more general Pickands’ representation

\(G_\eta(x)\) has Fréchet marginals with shape parameter \(\eta\) and scale parameter 1.

\[
V_\eta(x) = \int_{S_p} \eta \left\{ \max_{1 \leq i \leq p} \left( \frac{w_i}{x_i} \right) \right\}^{\frac{1}{\eta}} dH_\eta(w)
\]  

\[1 = \int_{S_p} \eta w_i^{-\eta} dH_\eta(w) = V_\eta(\infty, \cdots, \infty, 1, \infty, \cdots, \infty)
\]  

wherever the 1 is.
Multivariate representation of Ramos-Ledford-Tawn model

Define i.i.d $p$-dimensional random vectors $Z_k = (Z_{1,k}, \cdots, Z_{p,k})'$, $k = 1, \cdots, n$, with unit Fréchet marginals. By assuming multivariate regular variation, we have

$$
\bar{F}_{Z_1\cdots Z_p}(z_1, \cdots, z_p) = L(z_1, \cdots, z_p)z_1^{-c_1} \cdots z_p^{-c_p}
$$

(6)

where, $z_1, \cdots, z_p > 0$, $c_1 + \cdots + c_p = \frac{1}{\eta}$, $L$ is a multivariate slowly varying function (MSV) satisfying:

$$
\lim_{t \to \infty} \frac{L(tz_1, \cdots, tz_p)}{L(t, \cdots, t)} = g(z_1, \cdots, z_p)
$$

and $g(cz_1, \cdots, cz_p) = g(z_1, \cdots, z_p)$, $\forall c > 0, z_1, \cdots, z_p > 0$. 
Multivariate representation of Ramos-Ledford-Tawn model

Rewrite the Eq. (6), we get,

\[ \tilde{F}_{Z_1 \cdots Z_p}(z_1, \cdots, z_p) = \tilde{L}(z_1, \cdots, z_p)(z_1 \cdots z_p)^{-\frac{1}{p\eta}} \]  

(7)

where, \( \tilde{L} = L \cdot \left( \frac{z_1}{z_2} \right)^{\kappa_{12}/p} \left( \frac{z_2}{z_3} \right)^{\kappa_{23}/p} \cdots \left( \frac{z_1}{z_p} \right)^{\kappa_{1p}/p} \)

\( \kappa_{12} = c_2 - c_1, \kappa_{23} = c_3 - c_2, \cdots, \kappa_{1p} = c_p - c_1. \)

Define conditional variables:

\( (X_1, \cdots, X_p) = (Z_1/u, \cdots, Z_p/u) | (Z_1 > u, \cdots, Z_p > u), \) thus, \( X_1, \cdots, X_p \geq 1. \)

Then,

\[ \tilde{F}_{X_1 \cdots X_p}(x_1, \cdots, x_p) = \lim_{u \to \infty} \frac{\Pr(Z_1 > ux_1, \cdots, Z_p > ux_p)}{\Pr(Z_1 > u, \cdots, Z_p > u)} \]

\[ = \lim_{u \to \infty} \frac{\tilde{L}(ux_1, \cdots, ux_p)}{\tilde{L}(u, \cdots, u)} \frac{1}{(x_1 \cdots x_p)^{1/p\eta}} \]

\[ = \frac{g(x_1, \cdots, x_p)}{(x_1 \cdots x_p)^{1/p\eta}} = \frac{g^*(\frac{x_1}{\sum_{i=1}^p x_i}, \cdots, \frac{x_p-1}{\sum_{i=1}^p x_i})}{(x_1 \cdots x_p)^{1/p\eta}} \]  

(8)

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Multivariate Extreme Dependence and Financial Application
Multivariate representation of Ramos-Ledford-Tawn model

By taking a polar-coordinate transformation again, we have:

$$F_{X_1 \cdots X_p}(x_1, \cdots, x_p) = \int_{S_p} \eta \left\{ \min_{1 \leq i \leq p} \left( \frac{w_i}{x_i} \right) \right\}^{\frac{1}{\eta}} dH^*_\eta(w) \quad (9)$$

**Ramos-Ledford Constraint (general):** Suppose $x_i = u$, $i = 1, \cdots, p$, then, $F_{X_1 \cdots X_p}(u, \cdots, u) = u^{-\frac{1}{\eta}}$. Thus, the following normalising condition should hold:

$$1 = \int_{S_p} \eta \left\{ \min_{1 \leq i \leq p} w_i \right\}^{\frac{1}{\eta}} dH^*_\eta(w) = F_{X_1 \cdots X_p}(1, \cdots, 1) \quad (10)$$
From $h_0$ to $h_\eta$

**Theorem 1**: (A generalisation of Theorem 2 in Coles and Tawn (1991))
Given any positive function $h_0$ with finite $\frac{1}{\eta}$ moments on $S_p$, then,

$$h_\eta(w) = (m^\eta w)^{-(p + \frac{1}{\eta})} \prod_{i=1}^{p} m_i^\eta h_0\left( \frac{m_1^\eta w_1}{m^\eta w}, \cdots, \frac{m_p^\eta w_p}{m^\eta w} \right)$$

(11)

where,

$$m_i = \int_{S_p} \eta^1 u_i^1 h_0(u) \, du, \ i = 1, \cdots, p$$

(12)

is the density of a valid measure function $H_\eta$, satisfying the MEV constraint (general), i.e., Eq. (5).
From $h_\eta$ to $h^*_\eta$

**Theorem 2**: Given any function $h_\eta$ satisfying the MEV constraint (general), i.e., Eq. (5), then,

$$h^*_\eta(w) = \frac{h_\eta(w)}{\delta} \quad (13)$$

where,

$$\delta = p + (-1)^3 \sum V_\eta(\infty, \ldots, \infty, 1, \infty, \ldots, 1, \infty, \ldots, \infty) + \cdots + (-1)^{p+1} V_\eta(1, \ldots, 1) \quad (14)$$

is a valid density function of a measure function $H^*_\eta$, satisfying the Ramos-Ledford Constraint (general), i.e., Eq. (10).

*The special case when $d = 2$ and $\eta = 1$ is given in Ramos and Ledford (2008), where, $h^*(w) = h(w)/(2 - V(1, 1))$. 
From $h_\eta$ to $h^*_\eta$

**Sketch of the proof**: It is straightforward to check that,

\[
\int_{S_p} \eta \left\{ \min_{1 \leq i \leq p} w_i \right\} \frac{1}{\eta} h^*_\eta(w) \, dw = \int_{S_p} \eta \left\{ \min_{1 \leq i \leq p} w_i \right\} \frac{1}{\eta} h_\eta(w) \, dw = 1
\]

According to the inclusion-exclusion principle,

\[
\delta = \int_{S_p} \eta \left\{ \min_{1 \leq i \leq p} w_i \right\} \frac{1}{\eta} h_\eta(w) \, dw = (-1)^2 \sum_{i=1}^{p} \int_{S_p} \eta w_i \frac{1}{\eta} h_\eta(w) \, dw
\]

\[
+ (-1)^3 \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \int_{S_p} \eta \left\{ \max_{i \neq j, 1 \leq i \leq p, i+1 \leq j \leq p} (w_i, w_j) \right\} \frac{1}{\eta} h_\eta(w) \, dw
\]

\[
+ \cdots + (-1)^p \int_{S_p} \eta \left\{ \max_{1 \leq i \leq p} w_i \right\} \frac{1}{\eta} h_\eta(w) \, dw
\]
General survival functions

New $p$-dimensional joint survivors can be obtained from arbitrary measure $h_0$ (positive and satisfying that the integral in Eq. (12) is finite) on $S_p$ by the following equation:

$$
\bar{F}_{Z_1 \cdots Z_p}(z_1, \cdots, z_p) = \eta \lambda u_1^\eta \delta^{-1} \int_{S_p} \left\{ \min_{1 \leq i \leq p} \left( \frac{w_i}{z_i} \right) \right\}^{\frac{1}{\eta}} (m_1^\eta w^\eta)^{-(p+\frac{1}{\eta})} \prod_{i=1}^{p} m_i^{\eta} h_0\left( \frac{m_1^{\eta} w_1^{\eta}}{m_1^{\eta} w^{\eta}}, \cdots, \frac{m_p^{\eta} w_p^{\eta}}{m_1^{\eta} w^{\eta}} \right) dw
$$

(15)

where, as defined in Theorem 1 and 2,

$$
m_i = \int_{S_p} \eta u_i^{\eta} h_0(u) du, \quad i = 1, \cdots, p
$$

$$
\delta = p + (-1)^3 \sum \eta V_1(\infty, \cdots, \infty, 1, \infty, \cdots, \infty, 1, \infty, \cdots, \infty) + \cdots + (-1)^{p+1} V_1(1, \cdots, 1)
$$
Validity check

$\bar{F}_{Z_1Z_2}$ as a survival function is a valid survival function. As, for $x_1 \leq x_2, y_1 \leq y_2$, we can prove,

$$\bar{F}_{Z_1Z_2}(x_2, y_2) - \bar{F}_{Z_1Z_2}(x_1, y_2) - \bar{F}_{Z_1Z_2}(x_2, y_1) + \bar{F}_{Z_1Z_2}(x_1, y_1) = \lambda u \frac{1}{\eta} \delta^{-1} \left\{ \begin{array}{l} \frac{-1}{\eta} \int_{x_1}^{x_2} \eta w \frac{1}{\eta} h_\eta(w) dw - x_2 \frac{-1}{\eta} \int_{x_2}^{x_2+y_1} \eta w \frac{1}{\eta} h_\eta(w) dw \\ + \int_{y_1}^{y_2} \frac{-1}{\eta} \int_{x_1+y_1}^{x_2+y_1} \eta (1 - w) \frac{1}{\eta} h_\eta(w) dw \\ - y_2 \frac{-1}{\eta} \int_{x_1+y_2}^{x_2+y_2} \eta (1 - w) \frac{1}{\eta} h_\eta(w) dw \end{array} \right\} \geq 0$$

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Transformed Dirichlet model

Take the Dirichlet distribution for example, for \( d = 2 \),
\[
h_0(w) = \left[ \Gamma(\alpha_1)\Gamma(\alpha_2) \right]^{-1} \Gamma(\alpha \cdot 1) w^{\alpha_1 - 1} (1 - w)^{\alpha_2 - 1}
\]

According to the definition of \( m_i \),
\[
m_i = \eta \frac{\Gamma(\alpha \cdot 1)\Gamma(\alpha_i + \frac{1}{\eta})}{\Gamma(\alpha_i)\Gamma(\alpha \cdot 1 + \frac{1}{\eta})}, \quad i = 1, 2
\]

Then, based on Theorem 1,
\[
h_\eta(w) = \eta^{-1} \frac{\gamma_1 \gamma_2}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \Gamma(\alpha_1 + \alpha_2 + \frac{1}{\eta}) \left\{ \frac{\gamma_1 w + \gamma_2 (1 - w)}{\gamma_1 w + \gamma_2 (1 - w)} \right\}^{\alpha_1 - 1} \left\{ \frac{\gamma_2 (1 - w)}{\gamma_1 w + \gamma_2 (1 - w)} \right\}^{\alpha_2 - 1}
\]

where, \( \gamma = (\gamma_1, \gamma_2)' = \left( \frac{\Gamma(\alpha_1 + \frac{1}{\eta})}{\Gamma(\alpha_1)}, \frac{\Gamma(\alpha_2 + \frac{1}{\eta})}{\Gamma(\alpha_2)} \right)' \)
Transformed Dirichlet model

\[ V_\eta(x_1, x_2) = x_1^{-\frac{1}{\eta}} \left\{ 1 - \frac{\gamma_{1x_1}}{\gamma_x} (\alpha_1 + \frac{1}{\eta}, \alpha_2) \right\} + x_2^{-\frac{1}{\eta}} \frac{\gamma_{1x_2}}{\gamma_x} (\alpha_1, \alpha_2 + \frac{1}{\eta}) \]

where, \( l_\nu(a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^\nu t^{a-1}(1 - t)^{b-1} \, dt \) is the regularized incomplete beta function, and \( a, b > 0 \).

Thus,

\[ h^*_\eta(w) = \frac{\eta^{-1} \delta^{-1}_{\alpha_1, \alpha_2, \eta} \frac{\gamma_1\gamma_2}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{\Gamma(\alpha_1+\alpha_2+\frac{1}{\eta})}{\{\gamma_1 w + \gamma_2 (1-w)\}^{2+\frac{1}{\eta}}}}{\{\frac{\gamma_1 w}{\gamma_1 w + \gamma_2 (1-w)}\}^{\alpha_1-1} \{\frac{\gamma_2 (1-w)}{\gamma_1 w + \gamma_2 (1-w)}\}^{\alpha_2-1}} \]

\[ (17) \]

where, \( \delta_{\alpha_1, \alpha_2, \eta} = 1 + \frac{\gamma_{1x_1}}{\gamma_x} (\alpha_1 + \frac{1}{\eta}, \alpha_2) - \frac{\gamma_{1x_2}}{\gamma_x} (\alpha_1, \alpha_2 + \frac{1}{\eta}) \)
Transformed Dirichlet model

Then,

$$\bar{F}_{X_1X_2}(x_1, x_2) = \delta_{\alpha_1, \alpha_2, \eta}^{-1} x_1^{-\eta} I_{\gamma_{x_1}}(\alpha_1 + \frac{1}{\eta}, \alpha_2) + x_2^{-\eta} \left\{ 1 - I_{\gamma_{x_1}}(\alpha_1, \alpha_2 + \frac{1}{\eta}) \right\}$$

(18)

As $$\bar{F}_{Z_1Z_2}(z_1, z_2) = \lambda \bar{F}_{X_1X_2}(\frac{z_1}{u}, \frac{z_2}{u})$$, where $$\lambda = \Pr(Z_1 > u, Z_2 > u)$$. Then,

$$\bar{F}_{Z_1Z_2}(z_1, z_2; \Theta) = \lambda u^{-\eta} \delta_{\alpha_1, \alpha_2, \eta}^{-1} z_1^{-\eta} I_{\gamma_{x_1}}(\alpha_1 + \frac{1}{\eta}, \alpha_2) + z_2^{-\eta} \left\{ 1 - I_{\gamma_{x_1}}(\alpha_1, \alpha_2 + \frac{1}{\eta}) \right\}$$

(19)

where, parameter set $$\Theta = (\alpha_1, \alpha_2, \eta, \lambda)$$, $$\alpha_1, \alpha_2 > 0, 0 < \eta \leq 1, 0 \leq \lambda \leq 1.$$
Properties

Survival distribution plots \((\alpha_1, \alpha_2) = (2, 9), \lambda = 0.3, u = 0.5\)

Figure 1: Survival distribution, with \(\eta = 0.8\) and different pairs of \((\alpha_1, \alpha_2)\).

Asymmetric (symmetry occurs when \(\alpha_1 = \alpha_2\))

Convex \((\alpha_1 > 1, \alpha_2 > 1)\), Concave\((\alpha_1 < 1, \alpha_2 < 1)\) ·····
Likelihoods

Following the censored maximum likelihood estimation in Smith et al. (1997), the likelihood contribution of the region $R_{ij}; i = I(Z_1 > u), j = I(Z_2 > u)$ are respectively:

$$L_{11}(z_1, z_2) = \frac{\partial^2 \bar{F}_{Z_1 Z_2}(z_1, z_2)}{\partial z_1 \partial z_2}$$

$$= \lambda u^{\frac{1}{\eta}} \delta_{\alpha_1, \alpha_2, \eta} \frac{\gamma_1 \gamma_2 \Gamma(\alpha_1 + \alpha_2 + \frac{1}{\eta})}{\eta \Gamma(\alpha_1) \Gamma(\alpha_2)} \frac{(\gamma_1 z_1)^{\alpha_1 - 1}(\gamma_2 z_2)^{\alpha_2 - 1}}{(\gamma_1 z_1 + \gamma_2 z_2)^{\alpha_1 + \alpha_2 + \frac{1}{\eta}}}$$

$$L_{10}(z_1, z_2) = f(z_1) + \frac{\partial \bar{F}_{Z_1 Z_2}(z_1, u)}{\partial z_1}$$

$$= \exp(-1/z_1)z_1^{-2} + \lambda u^{\frac{1}{\eta}} \delta_{\alpha_1, \alpha_2, \eta} \left[-\frac{1}{\eta} \frac{1}{z_1^{\frac{1}{\eta} - 1}} \left\{ \frac{1}{\gamma_1 z_1 + \gamma_2 u^\eta} \left(\alpha_1 + \frac{1}{\eta}, \alpha_2\right) \right\} \right]$$

$$L_{01}(z_1, z_2) = f(z_2) + \frac{\partial \bar{F}_{Z_1 Z_2}(u, z_2)}{\partial z_2}$$

$$= \exp(-1/z_2)z_2^{-2} + \lambda u^{\frac{1}{\eta}} \delta_{\alpha_1, \alpha_2, \eta} \left[\frac{1}{\eta} z_2^{\frac{1}{\eta} - 1} \left\{ \frac{1}{\gamma_1 u^\eta + \gamma_2 z_2^{\eta}} \left(\alpha_1, \alpha_2 + \frac{1}{\eta}\right) - 1 \right\} \right]$$

$$L_{00}(z_1, z_2) = 2 \exp(-1/u) - 1 + \lambda$$
**Theoretical examples**

A. Asymptotic dependence - BEV logistic model \( r = 0.75 \) (TRUE VALUE: \( \eta = 1 \), symmetry \( \alpha_1 = \alpha_2 \))

Figure 2: Conditional density plots obtained from our distribution(red) at parameters \((0.15, 0.11, 0.99)\), Ramos-Ledford’s fitted distribution(green) and the true distribution of BEV logistic - \( r = 0.75 \)(dashed). Left shows the three plots and right shows only our’s and the true plot.
Theoretical examples

B. Positive association - Bivariate Normal dependence structure
\( \rho = 0.5 \) (TRUE VALUE: \( \eta = 0.75 \), symmetry \( \alpha_1 = \alpha_2 \))

Boundary case:
\[
\lim_{\alpha_1 \to 0} \bar{F}_{Z_1Z_2}(z_1, z_2) = \lambda u^{\frac{1}{\eta}} z_2^{-\frac{1}{\eta}}
\]
equivalently, when \( z_1 \) goes to \( \infty \).

Similarly,
\[
\lim_{\alpha_2 \to 0} \bar{F}_{Z_1Z_2}(z_1, z_2) = \lambda u^{\frac{1}{\eta}} z_1^{-\frac{1}{\eta}}
\]
Theoretical examples

**Figure 3:** Conditional density plots obtained from our distribution (red) at parameters (0.0000001, 0.0000001, 0.82), Ramos-Ledford’s fitted distribution (green) and the asymptotic distribution of BN - \( \rho = 0.5 \) as in Ledford and Tawn (1997) (dashed). Left shows the whole range of \( z_1 \) and right shows the range [100, 200].
Theoretical examples

C. Negative association - Bivariate Normal dependence structure

$\rho = -0.5 \text{ (TRUE VALUE: } \eta = 0.75, \text{ symmetry } \alpha_1 = \alpha_2)$

**Figure 4:** Conditional density plots obtained from our distribution (red) at parameters $(0.0000001, 0.0000001, 0.42)$, Ramos-Ledford’s fitted distribution (green) and the asymptotic distribution of BN - $\rho = -0.5$ as in Ledford and Tawn (1997) (dashed).

Poor performance in the negative association case?
**Assumption:** Stock indices of the banking industry best reflect the performance of banking systems.

**Definition:**
- banking crises occur when negative returns of the banking indices are over a certain high threshold $\tau$ (see Qin and Ren (2007));
- contagion occurs when banking crises occur in more than one countries or areas.

**Data:** daily banking indices for US, Euro area (EA, henceforth), Japan are from Thomson Reuters Datastream; those for China are from DAZHIHUI.

**Time horizon:** 01/02/2003-04/16/2008, 1255 sample points for each.
Research steps

**Step 1**: Compute the negative returns of each index $Y_i, i = 1, \cdots, 4$;

**Step 2**: Transform the returns into four new series $Z_i, i = 1, \cdots, 4$, which is unit Fréchet distributed;

**Step 3**: Determine the joint tail regions pairwisely, i.e., $\lambda_{ij}$ and $u_{ij}, i < j$;

- $u_{ij} < \min(\tau_i, \tau_j)$;
- $\tau_i, i = 1, \cdots, 4$ is set to be the 97.725th quantile of the single series fitted by generalised Pareto distribution (GPD) according to Qin and Ren (2007);
- WLOG, we set $\lambda_{ij} = 0.05$, $u_{ij}$ the 95th empirical quantile of $T_{ij} = \min(Z_i, Z_j)$.

**Step 4**: Diagnostic check on the parameters $\alpha_1, \alpha_2, \eta$ for the six pairs;

**Step 5**: Fit the transformed Dirichlet model and estimate the parameters by censored MLE;

**Step 6**: Calculate coincidence probability $p_{ij} = Pr(Z_i > \tau_i, Z_j > \tau_j)$ and contagion probability $p_{i|j} = \frac{Pr(Z_i > \tau_i, Z_j > \tau_j)}{Pr(Z_j > \tau_j)}$. 
## Results and conclusions

### Table 1: Coincidence and contagion probability matrix(%) 

<table>
<thead>
<tr>
<th></th>
<th>US</th>
<th>EA</th>
<th>Japan</th>
<th>China</th>
<th>US</th>
<th>EA</th>
<th>Japan</th>
<th>China</th>
</tr>
</thead>
<tbody>
<tr>
<td>US</td>
<td>-</td>
<td>1.27</td>
<td>.72</td>
<td>.88</td>
<td>-</td>
<td>79.9</td>
<td>45.2</td>
<td>55.3</td>
</tr>
<tr>
<td>EA</td>
<td>-</td>
<td>-</td>
<td>.82</td>
<td>.55</td>
<td>76.5</td>
<td>-</td>
<td>49.4</td>
<td>33.1</td>
</tr>
<tr>
<td>Japan</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>.46</td>
<td>25.8</td>
<td>29.4</td>
<td>-</td>
<td>16.5</td>
</tr>
<tr>
<td>China</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2.9</td>
<td>1.8</td>
<td>1.5</td>
<td>-</td>
</tr>
</tbody>
</table>
Results and conclusions

- US is the contagion origin of banking crises. The chance that the US banking crises propagate to China, EA and Japan are respectively 29 times, 1.13 times and 2.5 times of the reverse;

- The extreme dependence between US and EA is very strong and the coincidence probability of banking crises is very large. Banking crises occurring in the US almost surely have significant impact on the banking system in EA, and vice versa;

- The probability at which the banking crises in the US cause banking crises in China is much lower than the probability of banking crises propagate from US to EA, and higher that that from US to Japan. The chances that banking crises spread from EA or Japan to China are low;

- The probability that the US banking crises have strong impact on the Japanese banks are the lowest among all the three cases;

- Financial contagion is asymmetric between the two reverse directions.
Summary and future work

This study:

- proposes a general method to construct multivariate survival distributions which are able to model the joint tail with a wide class of dependence structure;
- derives a rich family of bivariate survival distributions which is at least a comparable alternative to that given by Ramos and Ledford (2008);
- applies the model to the area of financial contagion between China, US, Euro area and Japan banking systems.

In the future, we will come up with:

- more simulations
- \( d \geq 3 \)
References


-THE END-
Thank you!