

An Efficient and Incentive Compatible Ascending Auction for Multiple Complements¹

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Abstract: This paper proposes an ascending auction for efficiently allocating multiple complementary indivisible commodities. The seller has a reserve price for every bundle of goods and determines the quantity for sale based upon bidders' reported demands. The auctioneer announces a price for every bundle of goods, every bidder subsequently responds with a set of goods demanded at those prices, and then the auctioneer adjusts prices upwards for those over-demanded bundles and so on. First, we prove that the auction finds an efficient allocation and a market-clearing price for each bundle in finitely many steps, provided that all bidders bid sincerely or quasi-sincerely. Quasi-sincere bidding strategies can tolerate certain inaccurate or dishonest bidding behaviors of bidders. Second, when bidders are not assumed to be price-takers and can exercise their market power strategically, built on the above-mentioned auction we propose an incentive compatible dynamic procedure that always induces bidders to bid sincerely and finds an efficient allocation and a generalized Vickrey-Clarke-Groves payment for every bidder.

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1 Introduction

Our purpose is to provide an ascending-bid auction mechanism for efficiently allocating multiple complementary heterogeneous indivisible goods among many bidders. In the last twenty years or so dynamic auctions for the sale of multiple items have become phenomenally popular. For example, such auctions have been widely used by governments to sell spectrum licenses (see Klemperer (2004) and Milgrom (2004)), to procure goods and services, and to privatize state companies, and by firms to sell virtually all kinds of commodities via internet. Efficient and elegant auctions include those proposed by Kelso and Crawford (1982), Gul and Stacchetti (2000), Milgrom (2000), Perry and Reny (2005), and Ausubel (2004, 2006) among others. Auctions developed so far are most successful both in theory and in practice in handling the case when the goods for sale are substitutes. In contrast, how to deal with complementary goods via a dynamic auction has proved challenging. In a recent survey article, Maskin (2005) has called for designing a dynamic auction (like an English auction) for selling multiple complementary items, echoing Milgrom (2000), Jehiel and Moldovanu (2003), Noussair (2003), and Klemperer (2004) on similar problems. The current paper aims at providing a positive answer to this question.

There are several factors which make designing dynamic auction for complementary items difficult. First, unlike discrete markets in which a standard Walrasian equilibrium (i.e., the pricing rule is uniform and additive) is well known to exist when the goods are substitutes, it has not yet being clarified what notion of market-clearing prices (uniform prices or discriminatory prices) will be appropriate for the environment with complements. Third, as it is well known from Samuleson (1974) that there are a number of ways defining complements, then what kind of complementarity is suited for dynamic auction design? Fourth, as Milgrom (2000, 2004), Jehiel and Moldovanu (2003), and Klemperer (2004) point out, when there are strong complementarities between items, bidders may encounter the exposure problem in a simultaneous ascending auction which results in a loss of market efficiency. How to overcome the exposure problem? Fifth, in the current model, bidders, rather than being assumed to be price-takers, are allowed to exercise their market power strategically. And how to induce bidders to bid sincerely?

This paper attempts (1) to clarify the scope of the problem concerned with (indivisible) complements, and (2) to explore an appropriate solution for the environment, and (3) to devise an ascending auction that is tailored to the specific details of the context, assigns goods efficiently, and meanwhile induces bidders to bid truthfully. More specifically, we assume that every bidder views all goods as complements and has a private valuation on every bundle of goods and may have an incentive to economize on their private information. The complementarity (i.e., superadditivity) we consider is the most general one including gross complements and supermodularity as special cases. We demonstrate that superadditivity is sufficient for the existence of a nonlinear pricing Walrasian equilibrium (i.e., the pricing rule specifies a price for each bundle and is anonymous) but insufficient for the standard Walrasian equilibrium. More importantly, we propose a dynamic auction design for this environment. Roughly speaking, our basic dynamic auction works as follows. The auctioneer starts by announcing her reserve price for each bundle of items, every bidder subsequently responds with a set of items demanded at the current prices, and then the auctioneer adjusts prices upwards for those over-demanded bundles and so on. First, we prove that the auction discovers an efficient allocation and a market-clearing price for each bundle in finitely many rounds, provided that bidders bid sincerely or quasi-sincerely. Sincere-bidding strategies require every bidder to submit an optimal bundle relative to his utility function in every round of the auction, whereas quasi-sincere bidding strategies can tolerate certain inaccurate or dishonest bidding behaviors of bidders in the sense that every bidder needs to report an optimal bundle relative to his utility function only in every possible last round of the auction and can submit any bid in all other rounds as long as it does not give him a negative profit. Second, when bidders are permitted to bid strategically, we develop an incentive compatible dynamic procedure built upon the basic auction which always induces bidders to bid sincerely and finds an efficient allocation and a generalized Vickrey-Clarke-Groves (VCG) payment for every bidder.

It has long been recognized that ascending auctions have a number of important advantages over sealed-bid auctions. The famous VCG mechanism is a sealed-bid auction but is rarely used in practice despite its significant theoretical appeal; see e.g., Rothkopf, Teisberg, and Kahn (1990), Milgrom (2007) and Rothkopf (2007). Two major critiques of

the VCG auction are that (1) it requires all bidders to report their entire valuations over all possible bundles of goods, and (2) it may yield too low revenues for the seller. In reality, businessmen generally do not like to reveal their values or costs. Ascending auctions, however, require far less information from bidders and can keep the values of each winning bidder from exposing. In the language of Hurwicz (1973), they are privacy preserving and informationally efficient; see Ausubel (2004, 2006) and Perry and Reny (2005) for further discussions. Furthermore, as pointed out by Bergemann and Morris (2007), because ascending auctions create more transparent trading rules and offer opportunities for bidders to learn and adjust, they can also reduce payoff uncertainty and strategic uncertainty for bidders. Besides, the current auction has some additional merits. First, unlike the auctions of Gul and Stacchetti (2000), Ausubel (2006), Sun and Yang (2008A,B) which require every bidder to report all his optimal bundles in every round of the auction process, the current auction requires every bidder to report only one of his optimal bundles. Second, the current auction even allows bidders to bid quasi-sincerely and thus can tolerate them to bid inaccurately or dishonestly to some extent. Third, we permit the seller to have a reserve price for each bundle of goods. In the auction, the seller determines the quantity for sale based upon bidders' reported demands and will not sell any bundle if the bids are below its reserve price.

This paper closely relates to several papers. The basic idea of the price adjustment in the current auction is similar to that of Kelso and Crawford (1982) and our superadditivity condition can be seen as a counterpart of their gross substitutes condition but their notion of equilibrium is stronger than ours. Ausubel and Milgrom (2002) propose a family of ascending package auctions.⁴ Their auctions are designed to find a core allocation, which corresponds to a discriminatory and nonlinear pricing Walrasian equilibrium. Thus their notion of equilibrium is weaker than ours but their auctions were intended to deal with more complex situations. The current paper finds that a uniform price ascending auction is well suited for the environment where the goods being sold are complements. As argued by Milgrom (2004), discriminatory pricing fails to promote *the law of one price* and thus may be psychologically hard for some people to accept.⁵ Also Friedman (1991) and Miller

⁴A similar but sealed-bid package auction is due to Bernheim and Whinston (1986).

⁵However, discriminatory and nonlinear pricing is very natural for the situation where a seller is simul-

(1991) strongly advocate that uniform price auctions should be used where they can be used. Indeed, in real auctions, people are accustomed to uniform prices and in many European countries law even prohibits discriminatory prices among bidders.

Porter, Rassenti, Roopnarine and Smith (2003) investigate a combinatorial clock auction with uniform prices and discuss a number of practical issues concerning auction implementation. By exploring the methodology of experimental economics, they provide laboratory tests in support of the auction on several sets of spectrum licenses which exhibit strong complementarities. The current paper is more theoretical in the sense that we concentrate on designing an ascending auction, identifying existence conditions and examining the convergence of the auction. Ausubel (2004), Perry and Reny (2005) introduce efficient ascending auctions for identical goods in an interdependent value setting. Ausubel (2006) develops an ingenious incentive-compatible dynamic auction for selling heterogeneous substitutes. Sun and Yang (2008A,B) propose an incentive-compatible dynamic auction for a more general environment where there are two types of goods and bidders view goods of the same type as substitutes but goods across the two types as complements. The current paper deals with a different situation in which all goods are complements.

This paper is organized as follows. Section 2 sets up the model. Section 3 presents the basic ascending auction upon which the dynamic incentive compatible procedure is built in Section 4. Section 5 discusses an application. Finally, Section 6 concludes.

2 The Model

An auctioneer (seller) wishes to sell a set of heterogeneous indivisible goods (items) $N = \{1, 2, \dots, n\}$, to a group of potential bidders $M = \{1, 2, \dots, m\}$. Every bidder $i \in M$ attaches a monetary value (units of money) to each bundle of items, namely, each bidder i has a utility function $u^i : 2^N \rightarrow \mathbf{Z}_+$ with $u^i(\emptyset) = 0$, where 2^N denotes the family of all bundles of items. Every bidder is endowed with a sufficient amount of money in the sense that he can pay up to his value. The seller has a reserve price function $R : 2^N \rightarrow \mathbf{Z}_+$ with

taneously and independently negotiating with several buyers for important business, such as Boeing or Airbus sell their major products.

$R(\emptyset) = 0$. This means that if a bundle were sold to a bidder, the bidder would have to pay no less than the reserve price of the bundle. We impose the following mild assumptions on the model:

(A1) *Integer Private Values*: Every bidder i knows his own utility function $u^i : 2^N \rightarrow \mathbb{Z}_+$ privately and his utility function is integer-valued.

(A2) *Quasilinear Utility*: Every bidder i 's net profit is $u^i(S) - p(S)$ when he receives the bundle S and pays the price $p(S)$.

(A3) *Superadditivity for Bidders*: For every bidder i , and for any $A, B \in 2^N$ with $A \cap B = \emptyset$, bidder i 's utility function satisfies

$$u^i(A \cup B) \geq u^i(A) + u^i(B).$$

(A4) *Superadditivity for the Seller*: For for any $A, B \in 2^N$ with $A \cap B = \emptyset$, the seller's reserve price function R satisfies

$$R(A \cup B) \geq R(A) + R(B).$$

Assumptions (A1) and (A2) are standard and have been extensively used in the literature. Assumptions (A3) and (A4) say that both bidders and the seller view all items as complements in the sense that the value of combining two disjoint sets of items is no less than the sum of using both sets independently. This condition describes the most general complementarity among the items including both gross complements based on Marshallian demand, and supermodularity as special cases. Recall that a utility function $u^i : 2^N \rightarrow \mathbb{R}_+$ is said to be *supermodular* if for any $A, B \in 2^N$,

$$u^i(A \cup B) + u^i(A \cap B) \geq u^i(A) + u^i(B).$$

Supermodularity shows that a bidder's utility has increasing marginal returns. A utility function u^i is *submodular* if $-u^i$ is supermodular. Submodularity indicates that a bidder's utility has decreasing marginal returns. Superadditivity provides an alternative to the well-known gross substitutes (GS) condition of Kelso and Crawford (1982). The GS condition requires that every bidder regard all items as substitutes. This condition provides a general

sufficient condition for the existence of a standard Walrasian equilibrium and has been widely used in auction design; see e.g., Gul and Stacchetti (1999, 2000), Milgrom (2000) and Ausubel (2006).⁶ It is known that any utility function satisfying the GS condition must be submodular but a submodular utility function may violate the GS condition; see for example Gul and Stacchetti (1999) and Fujishige and Yang (2003).

Let 0 represent the seller and let $M_0 = M \cup \{0\}$ stand for all agents in the market. An allocation of items in N is a *partition* $\pi = (\pi(i), i \in M_0)$ of items among all agents in M_0 such that $\pi(i) \cap \pi(j) = \emptyset$ for all $i \neq j$ and $\cup_{i \in M_0} \pi(i) = N$. Note that $\pi(i) = \emptyset$ is allowed. An allocation π assigns the bundle $\pi(i)$ to agent i . If $\pi(0) \neq \emptyset$, then the bundle $\pi(0)$ is not sold and thus stays with the seller. Let \mathcal{A} denote the family of all allocations. An allocation π is *efficient* if for every allocation ρ we have

$$R(\pi(0)) + \sum_{i \in M} u^i(\pi(i)) \geq R(\rho(0)) + \sum_{i \in M} u^i(\rho(i)). \quad (2.1)$$

That is, an efficient allocation distributes the items to all agents in the most efficient way. Given an efficient allocation π^* , let $V(N) = \sum_{i \in M_0} u^i(\pi^*(i))$. We call $V(N)$ *the potential market value*. Clearly, this value is the same for any efficient allocation.

The following example shows that a standard Walrasian equilibrium may not exist in the presence of complementarity.

Example 1: Suppose that there are three bidders (1, 2, 3) and three items (A, B, C) in a market. Bidders' values and seller's reserve prices are given in Table 1. Clearly, all items are viewed by every bidder and the seller as complements. In this economy, there are two efficient allocations. We will show that none of these allocations can be supported by a price vector (p_A, p_B, p_C) . Consider, for instance, the efficient allocation $(\pi(1), \pi(2), \pi(3)) = (AB, C, \emptyset)$, where AB means the bundle that consists of items A and B . Suppose that the price vector (p_A, p_B, p_C) supports this allocation. Then for the seller, we must have $p_A \geq 1$, $p_B \geq 1$ and $p_C \geq 1$. For bidder 1, we must have $7 \geq p_A + p_B$. For bidder 2, we must have $p_C \leq 2$ and $p_A \geq 4$. For bidder 3, we must have $p_C \geq 2$ and

⁶Sun and Yang (2006A) provide an alternative but more general sufficient condition. In their reformulation, there are two types of goods. Goods of the same type are substitutes but goods across the two types are complements, as in the example of tables and chairs. Ostrovsky (2008) independently finds a similar condition.

$p_B + p_C \geq 6$. From $p_C = 2$, we have $p_B \geq 4$. Combining the inequalities $p_A \geq 4$ and $p_B \geq 4$ yields $p_A + p_B \geq 8$, which contradicts $7 \geq p_A + p_B$. Another efficient allocation (AB, \emptyset, C) can be shown in the same way. Thus, there exists no standard Walrasian equilibrium for this economy. Nevertheless, there exists a Walrasian equilibrium if the auctioneer adopts the following uniform nonlinear pricing rule $p(\emptyset) = 0$, $p(A) = 2$, $p(B) = 2$, $p(C) = 2$, $p(AB) = 6$, $p(AC) = 6$, $p(BC) = 6$, and $p(ABC) = 7$. For this pricing rule, the auctioneer can assign the bundle AB to bidder 1 by asking a price of 6 and the bundle C to bidder 2 by asking a price of 2. From the trading, the seller and bidder 1 gain net profit 8 and 1, respectively.

Table 1: Bidders' and seller's values over items.

	\emptyset	A	B	C	AB	AC	BC	ABC
Bidder 1	0	2	2	0	7	3	4	7
Bidder 2	0	2	0	2	3	6	3	6
Bidder 3	0	0	2	2	4	3	6	7
Seller	0	1	1	1	2	2	2	3

This example suggests that a possible way of developing a meaningful theory for coping with complementarity is to modify the standard notion of Walrasian equilibrium by adopting a uniform nonlinear pricing rule. A function $p : 2^N \rightarrow \mathbb{R}_+$ is a pricing function if $p(\emptyset) = 0$. A pricing function indicates a price $p(A)$ for each bundle A . A pricing function p is *linear* if for any bundle A , we can write $p(A) = \sum_{k \in A} p(\{k\})$. Otherwise, it is *nonlinear*. Nonlinear pricing has a long history in business practice and is widely used in many industries; see Wilson (1993), Armstrong (1996), and Png (2002). For a pricing function $p : 2^N \rightarrow \mathbb{R}_+$, bidder i 's demand correspondence $D^i(p)$, the net utility function $v^i(A, p)$, and the indirect utility function $V^i(p)$, are defined respectively by

$$\begin{aligned}
D^i(p) &= \{A \mid u^i(A) - p(A) = \max_{S \subseteq N} \{u^i(S) - p(S)\}\}, \\
v^i(A, p) &= u^i(A) - p(A), \text{ and} \\
V^i(p) &= \max_{A \subseteq N} \{u^i(A) - p(A)\}.
\end{aligned} \tag{2.2}$$

In nonlinear pricing settings, a seller has to consider not only which items to supply, but also how to supply them. For example, when the seller has two items for sale, she needs to decide whether she should supply the two items as two separate bundles or as a single bundle. In general, the seller can divide the items in N into $\max\{n, m+1\}$ disjoint bundles. Let $J = \{0, 1, \dots, \max\{n, m\}\}$ and $J_{-0} = J \setminus \{0\}$. Then for the seller every partition γ can be written as $\gamma = (\gamma(j), j \in J)$ such that $\gamma(j) \cap \gamma(k) = \emptyset$ for any different items $j, k \in J$ and $\cup_{j \in J} \gamma(j) = N$. In particular, if $\gamma(0) \neq \emptyset$, $\gamma(0)$ stands for the bundle that is not sold and thus stays with the seller. Let \mathcal{B} denote the family of all such partitions.

The seller's supply correspondence $S(p)$, and the revenue function $Re(p)$, are defined respectively by

$$\begin{aligned} S(p) &= \arg \max_{\gamma \in \mathcal{B}} \left\{ R(\gamma(0)) + \sum_{j \in J_{-0}} p(\gamma(j)) \right\}, \\ Re(p) &= \max_{\gamma \in \mathcal{B}} \left(R(\gamma(0)) + \sum_{j \in J_{-0}} p(\gamma(j)) \right). \end{aligned} \tag{2.3}$$

A supply set $\gamma \in S(p)$ is said to be an *allocation* when $R(\gamma(0)) + \sum_{i \in M} p(\gamma(i)) = Re(p)$ or equivalently $\cup_{i \in M(0)} \gamma(i) = N$.

Definition 2.1 *A nonlinear pricing Walrasian equilibrium consists of a price function $p^* : 2^N \rightarrow \mathbb{R}_+$ and an allocation π^* such that $\pi^*(i) \in D^i(p^*)$ for every bidder $i \in M$ and $\pi^* \in S(p^*)$.*

For a nonlinear pricing Walrasian equilibrium (p^*, π^*) , we call p^* an *equilibrium pricing function* and π^* an *equilibrium allocation*. We say that π^* is supported by p^* . Clearly, a nonlinear pricing Walrasian equilibrium (p^*, π^*) becomes the standard Walrasian equilibrium if p^* is linear. It is well known that the standard Walrasian equilibrium leads to the most efficient allocation of scarce resources. The following lemma shows that the notion of nonlinear pricing Walrasian equilibrium possesses the same appealing property.

Lemma 2.2 *If (p^*, π^*) be a nonlinear pricing Walrasian equilibrium, π^* is efficient. Furthermore, if ρ is an efficient allocation, (p^*, ρ) is also a nonlinear pricing Walrasian equilibrium.*

Some reader might worry that the concept of nonlinear pricing Walrasian equilibrium could be too general to be interesting. The following lemma and example are intended to dispel such concern.

Lemma 2.3 *Assume that Assumptions (A1) and (A2) hold for the auction model. If $\pi \in \mathcal{A}$ is an efficient allocation for which every $\pi(i)$ contains at most one item, then for every nonlinear equilibrium pricing function we can construct a linear equilibrium pricing function.*

The proof of the above two lemmas is relegated to the appendix.

Example 2: Assume that there are three bidders (1, 2, 3) and three items (A, B, C) in a market. The seller's reserve prices are all zero. Bidders' values over all bundles are given in Table 2. This example is taken from Bevia, Quinzii and Silva (1999), who show that there exists no (standard) Walrasian equilibrium in this market. We will prove that there does not exist any nonlinear pricing Walrasian equilibrium, either. Observe that $(\pi(1), \pi(2), \pi(3)) = (B, A, C)$ is the unique efficient allocation for which every $\pi(i)$ contains only one item. It follows immediately from Lemma 2.3 that there exists no nonlinear pricing Walrasian equilibrium.

Table 2: Bidders' values over items.

	\emptyset	A	B	C	AB	AC	BC	ABC
Bidder 1	0	10	8	2	13	11	9	14
Bidder 2	0	8	5	10	13	14	13	15
Bidder 3	0	1	1	8	2	9	9	10

Observe that the utility functions in this example are submodular. Therefore this example also demonstrates a striking fact that even if every bidder's utility function is submodular and thus exhibits decreasing marginal returns, a nonlinear pricing Walrasian equilibrium may still not exist. In contrast, the following equilibrium theorem shows that in spite of complementarity, there exists a nonlinear pricing Walrasian equilibrium. Surprisingly, the imposed conditions are remarkably general and easy to verify.

Theorem 2.4 *The auction model under Assumptions (A1)–(A4) has a nonlinear pricing Walrasian equilibrium.*

In the next section we will prove this theorem by designing a price adjustment process which actually finds a nonlinear pricing Walrasian equilibrium.

3 The Basic Dynamic Auction

In this section we first consider the basic case that every bidder bids sincerely, and then move to the case that every bidder bids quasi-sincerely. In the next section we allow bidders to bid strategically. In a dynamic auction, we say that bidder i bids sincerely with respect to utility function u^i if at every time $t \in \mathbb{Z}_+$ and any price function $p(t)$ at time t , he reports a bid $A_i(t) \in D^i(p(t)) = \{A \mid u^i(A) - p(t, A) = \max_{S \subseteq N} \{u^i(S) - p(t, S)\}\}$ with $A_i(t) = \emptyset$ when $\emptyset \in D^i(p(t))$.

3.1 The Sincere-Bidding Case

We now describe the main idea of our basic dynamic auction. The design of this auction process is based on the classical tatonnement process with several modifications and is quite similar to the salary-adjustment process of Kelso and Crawford (1982). The auctioneer calls out a price vector p that specifies a price for each bundle. Every bidder i responds by reporting a demand bundle $A_i \in D^i(p)$. The auctioneer then examines the aggregated reported demands and decides her supply of items. If a bundle is over-demanded, the auctioneer raises its price but he will not change the price of any bundle which is balanced or over-supplied.⁷ This process stops at some price vector p^* when no bundle is over-demanded. Then according to bidders' reported demands and some rules to be specified below, the auctioneer assigns a reported bundle B to each bidder who is asked to pay the current corresponding price $p^*(B)$.

Given a pricing function $p : 2^N \rightarrow \mathbb{R}_+$, each bidder $i \in M$ reports a demand bundle $A_i \in D^i(p)$ and the seller chooses a supply set $\gamma \in S(p)$. With respect to the sets γ and A_i , we say that a bundle $A \in 2^N \setminus \{\emptyset\}$ is *over-demanded* if it is demanded by more than one bidder (i.e., $A_i = A_j = A$ for at least two bidders i, j) or demanded by some bidder i (i.e., $A_i = A$) but his bundle A_i is not in the seller's supply set γ (i.e., $A_i \notin \gamma$).⁸ A nonempty bundle A in γ is *over-supplied* if its price $p(A)$ is strictly greater than its reserve price $R(A)$

⁷This is quite different from the classical tatonnement process which requires the reduction of the price of an over-supplied good.

⁸This definition of over-demanded sets which depends on the seller's supply set is a crucial and novel point for the price adjustment of the current auction.

and no bidder demands it (i.e., there is no bidder i whose demand A_i equals A). Keep in mind that an over-demanded or over-supplied bundle must be nonempty and the empty set can be demanded by many bidders without any cost. Having all the preparations, we can now give a formal description of the price adjustment process.

The basic auction process

Step 1: The auctioneer announces an initial pricing function $p(0) : 2^N \rightarrow \mathbb{Z}_+$ so that $p(0, S)$ equals the reserve price $R(S)$ for every bundle $S \subseteq N$. Set $t := 0$ and go to Step 2.

Step 2: At each round $t = 0, 1, 2, \dots$, every bidder i reports a sincere bid $A_i(t)$ at the prices $p(t)$. Then the auctioneer chooses a supply set $\gamma(t) \in S(p(t))$ so that the market yields the least over-demanded bundles. If no bundle is over-demanded at $p(t)$, go to Step 3. But if there is an over-demanded bundle, the auctioneer raises the price of each over-demanded bundle by one unit but holds the price of any other bundle unchanged. Set $t := t + 1$ and return to Step 2.

Step 3: The auctioneer assigns the bundle $A_i(t)$ to bidder i who is asked to pay the price $p(t, A_i(t))$ in return, and in addition for any nonempty bundle $B \in \gamma(t)$ which is not demanded by any bidder at $p(t)$, the auctioneer assigns the bundle to the seller if $p(t, B) = R(B)$, otherwise, the auctioneer assigns the bundle to some bidder⁹ who previously demanded the bundle but was the last to give up, and who is asked to pay $p(t, B)$. Then the process stops.

Several remarks on the auction process are in order. First, because the sequence of generated prices for each bundle is never decreasing even if a bundle is over-supplied, the auction can be viewed as an ascending bid procedure. Second, to accelerate the auction process and prevent the prices from being driven too high, the auctioneer is required to report a supply set $\gamma(t)$ so that the market yields the least over-demanded bundles, i.e., the number of over-demanded bundles attains minimum. Thus, if in some round having some supply set γ which yields no over-demanded bundles, the auctioneer will choose such

⁹If there are several such bidders, the auctioneer randomly chooses one.

a supply set. Third, in the final round of the process (Step 3) the auctioneer assigns the bundle $A_i(t)$ to bidder i and then may assign an additional bundle B to this same bidder if the bidder once demanded the bundle, and asks the bidder to pay the sum of $p(t, A)$ and $p(t, B)$, whereas all unsold bundles are assigned to the seller. This operation will be called *the complementary activity rule*, which is a novel feature of the current process. It will be shown that $p(t, A) + p(t, B)$ is actually equal to $p(t, A \cup B)$, a crucial fact for this auction process.

We also point out that in the current auction process, every bidder i needs to report only one bundle $A_i(t)$ from his demand correspondence $D^i(p(t))$ in each round t , whereas in those of Gul and Stacchetti (2000), Ausubel (2006), Sun and Yang (2008A, B), every bidder i needs to report his entire demand correspondence $D^i(p(t))$. Kelso and Crawford (1982) and Milgrom (2000) require every bidder i to submit only one bundle $A_i(t)$ in each round but cannot guarantee to find an equilibrium in finitely many rounds.

Before proving the convergence of the auction, let us illustrate through Example 1 how the auction actually operates. The auctioneer starts with the reserve prices $p(0) = (1, 1, 1, 2, 2, 2, 3)$ for all bundles and chooses a supply set $\gamma(0) = (\emptyset, AB, C)$ and bidder 1 bids AB , bidder 2 bids AC , bidder 3 bids BC . At $p(0)$, the bundles AC and BC are over-demanded and so the auctioneer adjusts the price vector to $p(1)$ by increasing the price of AC and BC each by one unit and keeping the prices of all other bundles unchanged. At $p(1)$, the auctioneer supplies $\gamma(1) = (\emptyset, AC, B)$ and bidder 1 demands AB , bidder 2 demands AC but bidder 3 demands ABC . Note that at $p(1)$ both AB and ABC are over-demanded. The auctioneer adjusts the price vector to $p(2)$. After 7 rounds, the auction proceeds to the price vector $p(8)$ at which bidder 1 bids AB , bidder 2 demands nothing and bidder 3 bids B . Note that at $p(8)$ bidder 2 actually has several optimal bundles (including the empty set) to bid but is required to report the empty set. The auction stops at $p(9)$ for which no bundle is over-demanded. By the auction rule, bidder 1 gets AB and pays 6, bidder 3 gets nothing and pays nothing, but bidder 2 is assigned with C and is asked to pay 2 (note that C is an optimal bundle of bidder 2 at $p(9)$!), since bidder 2 demanded C at round 5 and was the last person to give up C . So the process ends with the efficient allocation $\pi^* = (\pi(1), \pi(2), \pi(3)) = (AB, C, \emptyset)$. Clearly, $(p(9), \pi^*)$

is a nonlinear pricing Walrasian equilibrium. The whole process of demands and supplies at the adjusted price vectors is shown in Table 3, where the price vector is understood as $p = (p(A), p(B), p(C), p(AB), p(AC), p(BC), p(ABC))$ with $p(\emptyset) = 0$, seller's column indicates her supplies $\gamma = (\gamma(0), \gamma(1), \gamma(2))$ with $\gamma(3) = \emptyset$ being ignored, whereas each bidder's column indicates his demands.

Table 3: Demands and supplies at the adjusted price vectors for Example 1.

Price Vector	Seller	Bidder 1	Bidder 2	Bidder 3
$p(0) = (1, 1, 1, 2, 2, 2, 3)$	(\emptyset, AB, C)	AB	AC	BC
$p(1) = (1, 1, 1, 2, 3, 3, 3)$	(\emptyset, AC, B)	AB	AC	ABC
$p(2) = (1, 1, 1, 3, 3, 3, 4)$	(\emptyset, AC, B)	AB	AC	ABC
$p(3) = (1, 1, 1, 4, 3, 3, 5)$	(\emptyset, AB, C)	AB	AC	BC
$p(4) = (1, 1, 1, 4, 4, 4, 5)$	(\emptyset, AB, C)	AB	AC	BC
$p(5) = (1, 1, 1, 4, 5, 5, 5)$	(\emptyset, AC, B)	AB	C	ABC
$p(6) = (1, 1, 2, 5, 5, 5, 6)$	(\emptyset, AB, C)	AB	A	ABC
$p(7) = (2, 1, 2, 5, 5, 5, 7)$	(\emptyset, AB, C)	AB	AC	BC
$p(8) = (2, 1, 2, 5, 6, 6, 7)$	(\emptyset, A, BC)	AB	\emptyset	B
$p(9) = (2, 2, 2, 6, 6, 6, 7)$	(\emptyset, AB, C)	AB	\emptyset	\emptyset

Theorem 3.1 *Suppose that Assumptions (A1)–(A4) hold for the auction model. When every bidder bids sincerely, the basic auction process yields a nonlinear pricing Walrasian equilibrium, in a finite number of rounds.*

Proof: It is easy to see that the auction process stops at some step t^* , because as soon as the price of any (nonempty) bundle becomes higher than any bidder's valuation for it, no bidder will demand it. Note that the price of the empty set is always fixed at zero.

For ease of notation, let $p^* = p(t^*)$ and let $A_i^* = A_i(t^*)$ that is demanded by bidder i , and let $\gamma^* = \gamma(t^*) \in S(p^*)$ that is the supply set of the seller. Recall that by definition γ^* is a seller's partition of all the items N . We will construct an allocation π^* so that (p^*, π^*) constitutes a nonlinear pricing Walrasian equilibrium. Note that at p^* , no (nonempty)

bundle is over-demanded. Thus, for any bidder $i \in M$, if his demand bundle A_i^* is not empty, it must be in the supply set γ^* . Moreover, for any two bidders $i, l \in M$, with $A_i^* \neq \emptyset$ and $A_l^* \neq \emptyset$, we must have $A_i^* \cap A_l^* = \emptyset$. If $\cup_{i \in M} A_i^* = N$, let $\pi^*(i) = A_i^*$, then clearly (p^*, π^*) is a nonlinear pricing Walrasian equilibrium and we are done.

Suppose otherwise that there is some nonempty bundle B in the supply set γ^* which is not demanded by any bidder. Such a bundle is called a *squeezed out bundle*. We first consider the case in which $p^*(B) = R(B)$. Let $(\gamma^*(i), i \in K)$ be the collection of all such bundles. Note $\gamma^*(i) \cap \gamma^*(l) = \emptyset$ for $i, l \in K$ and $i \neq l$. Let $\pi^*(0) = \cup_{i \in K} \gamma^*(i)$. We assign $\pi^*(0)$ to the seller. By the Superadditivity Assumption (A4), we know that

$$R(\pi^*(0)) \geq \sum_{i \in K} R(\gamma^*(i)).$$

Next, we consider the case in which $p^*(B) > R(B)$. This implies that the bundle B was demanded by some bidder at some step. Let t be the last step at which B is demanded by some bidder l . By the auction rule B can be assigned to bidder l who is asked to pay the current price $p^*(B)$. We will show that bidder l loses nothing in having the bundle B and paying the price. By the auction rule and Assumption (A2), we must have $V^l(p(t)) = v^l(B, p(t)) = u^l(B) - p(t, B) \geq 1$, and $p^*(B) = p(t, B)$ or $p^*(B) = p(t, B) + 1$. For the bidder l , it holds that

$$v^l(B, p^*) = u^l(B) - p^*(B) \geq 0. \quad (3.4)$$

We need to consider the following two situations.

Case 1. When $A_l^* = \emptyset$, let $\pi^*(l) = B$. Because $A_l^* \in D^l(p^*)$ and $A_l^* = \emptyset$, we have $V^l(p^*) = 0 \geq v^l(B, p^*)$. Recall that $v^l(B, p^*) \geq 0$. These inequalities lead to $v^l(B, p^*) = 0$, which implies $\pi^*(l) \in D^l(p^*)$.

Case 2. When $A_l^* \neq \emptyset$, let $\pi^*(l) = A_l^* \cup B$. For the seller, we know that

$$p^*(A_l^*) + p^*(B) \geq p^*(\pi^*(l)). \quad (3.5)$$

For the bidder l , the Superadditivity Assumption (A3) implies that

$$u^l(\pi^*(l)) \geq u^l(A_l^*) + u^l(B). \quad (3.6)$$

It follows from (3.5) and (3.6) that

$$\begin{aligned}
u^l(\pi^*(l)) - p^*(\pi^*(l)) &\geq u^l(\pi^*(l)) - (p^*(A_l^*) + p^*(B)) \\
&\geq u^l(A_l^*) - p^*(A_l^*) + u^l(B) - p^*(B) \\
&\geq u^l(A_l^*) - p^*(A_l^*)
\end{aligned}$$

where the last inequality is derived from (3.4). Because $A_l^* \in D^l(p^*)$, we have $\pi^*(l) \in D^l(p^*)$ (i.e., $\pi^*(l)$ is also an optimal bundle of bidder l at p^*). Consequently, it further implies that

$$u^l(\pi^*(l)) - p^*(\pi^*(l)) = u^l(\pi^*(l)) - (p^*(A_l^*) + p^*(B)) = u^l(A_l^*) - p^*(A_l^*)$$

which leads to

$$p^*(\pi^*(l)) = p^*(A_l^*) + p^*(B). \quad (3.7)$$

So in both cases bidder l loses nothing from having the nonessential bundle B and paying the price $p^*(B)$. As a result, the indirect utility of bidder l remains unchanged.

We can repeat this adjustment until every such squeezed out bundle B (i.e., $p^*(B) > R(B)$) in γ^* is assigned to some bidder. For any bidder i who is not assigned with any squeezed out bundle, let $\pi^*(i) = A_i^*$. So in the end each bidder i gets a bundle $\pi^*(i)$ in his demand set. Because γ^* is a seller's partition of N , $(\pi^*(0), \dots, \pi^*(m))$ must be an allocation of N . Furthermore, it follows from the formula (3.7) that $R(\pi^*(0)) + \sum_{i \in M} p^*(\pi^*(i)) = R(\gamma^*(0)) + \sum_{j \in J_{-0}} p^*(\gamma^*(j)) = Re(p^*)$. That is, the allocation $\pi^* \in S(p^*)$. Consequently, (p^*, π^*) is a nonlinear pricing Walrasian equilibrium and we are done. \square

As a consequence of Theorem 3.1, we have given a constructive proof for Theorem 2.4.

To accelerate the basic auction process, we introduce a price function operator. For any given price function $p : 2^N \rightarrow \mathbb{R}_+$, we define a superadditive price function $T(p)$ as follows: For every bundle $S \in 2^N$,

$$T(p)(S) = \max_{\gamma \in \mathcal{B}(S)} \left(\sum_{j \in J_{-0}} p(\gamma(j)) \right), \quad (3.8)$$

where $\mathcal{B}(S)$ denotes the family of all partitions of the elements in S . Let \mathcal{P} be the space of all price functions $p : 2^N \rightarrow \mathbb{R}_+$, and \mathcal{P}_{SA} be the space of all superadditive price functions $p : 2^N \rightarrow \mathbb{R}_+$. Then, T defines a function mapping from \mathcal{P} to \mathcal{P}_{SA} . Now we have the following useful observation.

Lemma 3.2 *If (p, π) is a nonlinear pricing Walrasian equilibrium, then $(T(p), \pi)$ is also a nonlinear pricing Walrasian equilibrium.*

Proof: Since $\pi \in S(p)$, by the definition of supply set, we have

$$p(\pi(i)) = \max_{\gamma \in \mathcal{B}(\pi(i))} \left(\sum_{j \in J_{-0}} p(\gamma(j)) \right)$$

for every $i \in M(0)$, i.e., $T(p(\pi(i))) = p(\pi(i))$ for all $i \in M(0)$. Similarly, it can be shown that $\pi \in S(T(p))$. Moreover, observe that $T(p)(S) \geq p(S)$ for any bundle $S \subseteq N$, we see $\pi(i) \in D^i(T(p))$ for every $i \in M$. This shows that $(T(p), \pi)$ is indeed a nonlinear pricing Walrasian equilibrium. \square

In each round t of the basic auction process, according to the price adjusting rule given in Step 2, we obtain the new price function $p(t+1)$ from $p(t)$. Now we can set $p(t+1) := T(p(t+1))$ for the next round $t+1$ and modify also the operation of the auctioneer slightly as discussed in Subsection 3.2. The reason why we can use the operator T is quite simple. Consider two disjoint bundles A and B and their union $A \cup B$. Suppose that at the round t , $p(A) + p(B) > p(A \cup B)$. In this case, even if the bundle $A \cup B$ is over-demanded, the seller will not choose this bundle to supply. She may supply the bundle only if $p(A \cup B) = p(A) + p(B)$. The operator T sets precisely $T(p(A \cup B)) = p(A) + p(B)$ and therefore expedites the adjustment process by avoiding many possible steps between $p(A) + p(B)$ and $p(A \cup B)$.

3.2 The Quasi-Sincere Bidding Case

In practice, bidders may not always be able to submit a precise optimal demand bundle in every step of an auction. In the following we will investigate to what extent the current auction can tolerate inaccurate bidding behavior of bidders. In a dynamic auction, we say that bidder i *bids quasi-sincerely* with respect to utility function u^i if he reports a bid $A_i(t^*) \in D^i(p(t^*)) = \{A \mid u^i(A) - p(t^*, A) = \max_{S \subseteq N} \{u^i(S) - p(t^*, S)\}\}$ (with $A_i(t^*) = \emptyset$ when $\emptyset \in D^i(p(t^*))$) in every possible last round $t = t^*$ of the auction, and always reports an empty bundle or a nonempty bundle $A_i(t)$ which satisfies $u^i(A_i(t)) - p^i(t, A_i(t)) \geq 1$ for all other rounds t before the termination of the auction. In other words, quasi-sincere

bidding requires bidder i to be accurate only in every possible last round of the auction and allows him to submit any bid in any other rounds as long as it gives him a positive profit or it is an empty bundle when it gives him a zero profit.

Because bidders usually do not know whether the current round is a possible last round or not, we need to modify the basic auction process by requiring that when the auctioneer finds there is no over-demanded bundle, she reminds every bidder that “this may be the last round, please report your optimal demand bundle”, and so she gives the bidders a chance to revise their bids. Both ‘quasi-sincere bidding’ and ‘the auctioneer reminding the bidders in every possible last round’ imitate what happens in real auction houses where bidders do not submit bids in every round of the auction and the auctioneer does remind the bidders in the last round by e.g., hitting a hammer several times. Formally, we can modify the previous basic auction process as follows:

The modified basic auction process

Step 1: The auctioneer chooses an initial pricing function $q(0) : 2^N \rightarrow \mathbb{Z}_+$ so that $q(0, S)$ equals the reserve price $R(S)$ for every bundle $S \subseteq N$. Set $t := 0$ and go to Step 2.

Step 2: At each round $t = 0, 1, 2, \dots$, the auctioneer announces $p(t) = T(q(t))$, and every bidder i reports quasi-sincerely a bid $A_i(t)$ at the announced prices $p(t)$. Then the auctioneer selects a supply set $\gamma(t) \in S(p(t))$ such that for every non-empty supply bundle $B \in \gamma(t)$ with $p(t, B) > R(B)$, there exists some bidder j whose bid $A_j(t')$ at some round $t' \leq t$ equals B at price $p(t', B) = p(t, B)$ or $p(t, B) - 1$, and such that the market yields the least over-demanded bundles. If no bundle is over-demanded at $p(t)$, go to Step 3. But if there is an over-demanded bundle, the auctioneer raises the price of each over-demanded bundle by one unit but holds the price of any other bundle unchanged, and obtains a new price $q(t + 1)$. Set $t := t + 1$ and return to Step 2.

Step 3: The auctioneer reminds all bidders that “this may be the last round, please report your optimal demand bundle”. Then, every bidder i resubmits sincerely a

new bid $A_i(t) \in D^i(p(t))$ when his former bid $A_i(t)$ is not an optimal bid. If no bidder revises his bid, go to Step 4. Otherwise, if some bidder revises his bid, the auctioneer selects a supply set $\gamma(t) \in S(p(t))$ such that for every non-empty supply bundle $B \in \gamma(t)$ with $p(t, B) > R(B)$, there exists some bidder j whose bid $A_j(t')$ at some round $t' \leq t$ equals B at price $p(t', B) = p(t, B)$ or $p(t, B) - 1$, and such that the market yields the least over-demanded bundles. If no bundle is over-demanded at $p(t)$, go to Step 4. But if there is an over-demanded bundle, the auctioneer raises the price of each over-demanded bundle by one unit but holds the price of any other bundle unchanged, and obtains a new price vector $q(t+1)$. Set $t := t+1$ and return to Step 2.

Step 4: The auctioneer assigns the bundle $A_i(t)$ to bidder i who is asked to pay the price $p(t, A_i(t))$ in return, and in addition for any nonempty bundle $B \in \gamma(t)$ which is not demanded by any bidder at $p(t)$, the auctioneer assigns the bundle to the seller if $p(t, B) = R(B)$, otherwise, the auctioneer assigns the bundle to some bidder¹⁰ who previously demanded the bundle but was the last to give up, and who is asked to pay $p(t, B)$. Then the process stops.

Theorem 3.3 *Suppose that Assumptions (A1)–(A4) hold for the auction model. When every bidder bids quasi-sincerely, the modified basic auction process yields a nonlinear pricing Walrasian equilibrium, in a finite number of rounds.*

Proof: Suppose the modified auction stops at some step t^* . Let $p^* = p(t^*)$, $A_i^* = A_i(t^*)$, and $\gamma^* = \gamma(t^*) \in S(p^*)$. Then, $A_i^* \in D^i(p^*)$ for every $i \in M$ and $\gamma^* \in S(p^*)$. We will construct an allocation π^* so that (p^*, π^*) constitutes a nonlinear pricing Walrasian equilibrium. Note that for every bidder $i \in M$, if his demand bundle A_i^* is not empty, it must be in the supply set γ^* . Moreover, for any two bidders $i, l \in M$, with $A_i^* \neq \emptyset$ and $A_l^* \neq \emptyset$, we must have $A_i^* \cap A_l^* = \emptyset$. If $\cup_{i \in M} A_i^* = N$, let $\pi^*(i) = A_i^*$, then clearly (p^*, π^*) is a nonlinear pricing Walrasian equilibrium and we are done.

Suppose otherwise that there is some nonempty bundle B in the supply set γ^* which is not demanded by any bidder in the last step. Such a bundle is said to be *squeezed out*.

¹⁰If there are several such bidders, the auctioneer randomly chooses one.

We first consider the case in which $p^*(B) = R(B)$. Let $(\gamma^*(i), i \in K)$ be the collection of all such bundles. Note $\gamma^*(i) \cap \gamma^*(l) = \emptyset$ for $i, l \in K$ and $i \neq l$. Let $\pi^*(0) = \cup_{i \in K} \gamma^*(i)$. We assign $\pi^*(0)$ to the seller. By the Superadditivity Assumption (A4), we know that

$$R(\pi^*(0)) \geq \sum_{i \in K} R(\gamma^*(i)).$$

Next, we consider the case in which $p^*(B) > R(B)$. This implies that the bundle B was demanded by some bidder at some step. Let t be the last step at which B is demanded at price $p(t) = p^*(B)$ or $p^*(B) - 1$ by some bidder l . By the auction rule B can be assigned to bidder l who is asked to pay the current price $p^*(B)$. We will show that bidder l loses nothing in having the bundle B and paying the price. Since bidder l bids quasi-sincerely, we have $v^l(B, p(t)) = u^l(B) - p(t, B) \geq 1$. Notice that $p^*(B) = p(t, B)$ or $p^*(B) = p(t, B) + 1$. Hence, for the bidder l , it holds that

$$v^l(B, p^*) = u^l(B) - p^*(B) \geq 0. \quad (3.9)$$

We need to consider the following two situations.

Case 1. When $A_l^* = \emptyset$, let $\pi^*(l) = B$. Because $A_l^* \in D^l(p^*)$ and $A_l^* = \emptyset$, we have $V^l(p^*) = 0 \geq v^l(B, p^*)$. Recall that $v^l(B, p^*) \geq 0$. These inequalities lead to $v^l(B, p^*) = 0$, which implies $\pi^*(l) \in D^l(p^*)$.

Case 2. When $A_l^* \neq \emptyset$, let $\pi^*(l) = A_l^* \cup B$. For the seller, we know that

$$p^*(A_l^*) + p^*(B) \geq p^*(\pi^*(l)). \quad (3.10)$$

For the bidder l , the Superadditivity Assumption (A3) implies that

$$u^l(\pi^*(l)) \geq u^l(A_l^*) + u^l(B). \quad (3.11)$$

It follows from (3.10) and (3.11) that

$$\begin{aligned} u^l(\pi^*(l)) - p^*(\pi^*(l)) &\geq u^l(\pi^*(l)) - (p^*(A_l^*) + p^*(B)) \\ &\geq u^l(A_l^*) - p^*(A_l^*) + u^l(B) - p^*(B) \\ &\geq u^l(A_l^*) - p^*(A_l^*) \end{aligned}$$

where the last inequality is derived from (3.9). Because $A_l^* \in D^l(p^*)$, we have $\pi^*(l) \in D^l(p^*)$ (i.e., $\pi^*(l)$ is also an optimal bundle of bidder l at p^*). Consequently, it further implies that

$$u^l(\pi^*(l)) - p^*(\pi^*(l)) = u^l(\pi^*(l)) - (p^*(A_l^*) + p^*(B)) = u^l(A_l^*) - p^*(A_l^*)$$

which leads to

$$p^*(\pi^*(l)) = p^*(A_l^*) + p^*(B). \quad (3.12)$$

So in both cases bidder l loses nothing from having the nonessential bundle B and paying the price $p^*(B)$. As a result, the indirect utility of bidder l remains unchanged.

We can repeat this adjustment until every such squeezed out bundle B (i.e., $p^*(B) > R(B)$) in γ^* is assigned to some bidder. For any bidder i who is not assigned with any squeezed out bundle, let $\pi^*(i) = A_i^*$. So in the end each bidder i gets a bundle $\pi^*(i)$ in his demand set. Because γ^* is a seller's partition of N , $(\pi^*(0), \dots, \pi^*(m))$ must be an allocation of N . Furthermore, it follows from the formula (3.12) that $R(\pi^*(0)) + \sum_{i \in M} p^*(\pi^*(i)) = R(\gamma^*(0)) + \sum_{j \in J_{-0}} p^*(\gamma^*(j)) = Re(p^*)$. That is, the allocation $\pi^* \in S(p^*)$. Consequently, (p^*, π^*) is a nonlinear pricing Walrasian equilibrium and we are done. \square

4 The Dynamic Incentive Compatible Auction

In the previous section we show that when the bidders bid (quasi-)sincerely in the (modified) basic auction, an exact nonlinear pricing Walrasian equilibrium will be found in a finite number of steps. A natural and fundamental question arises whether one could expect the buyers to bid sincerely, because buyers' valuations are private information and unobservable, it is quite conceivable that buyers may bid whatever they like as long as it serves their interest and does not openly violate the rules. Based upon the basic auction process, we shall design an efficient and incentive compatible dynamic auction which induces every bidder to bid sincerely and yields an efficient allocation and a generalized VCG payment for every bidder in the same environment as in Section 3.

Let \mathcal{M} denote the auction market with the set M of bidders and the set N of items, and for each bidder $i \in M$, let \mathcal{M}_{-i} denote the market \mathcal{M} without bidder i . Let $M_{-i} = M \setminus \{i\}$ for every $i \in M$ and for convenience also let $\mathcal{M}_{-0} = \mathcal{M}$, $M_{-0} = M$ and $M(0) = M \cup \{0\}$. Furthermore, assume that the auctioneer knows some integer value U^* greater than any bidder's possible maximum value, and let \mathcal{U} denote the family of all superadditive utility functions $u : 2^N \rightarrow \mathbf{Z}_+ \cap [0, U^*]$. We now describe the dynamic auction game of incomplete

information in which every bidder behaves strategically and treats his utility function as private information. In this environment, each bidder $i \in M$ is informed only of his own utility function $u^i \in \mathcal{U}$ and of a joint probability distribution $F(\cdot)$ from which the profile $\{u^i\}_{i \in M}$ is drawn with $u^i \in \mathcal{U}$. Note that the reserve price function R of the seller is also drawn from \mathcal{U} and will reveal to the bidders when the auction starts.

Several price functions will be used in our strategy-proof dynamic auction. In each round $t \in \mathbb{Z}_+$ of the auction, $p^0(t)$ denotes *the open or second price function* which is announced by the auctioneer, and $p^i(t)$ stands for *the price function faced by bidder $i \in M$* . Meanwhile, the auctioneer (or the seller) has her own price function $p^{-l}(t)$ for each market \mathcal{M}_{-l} , $l \in M(0)$. We call $p^{-0}(t)$ *the first price function* in round t . The auction works roughly as follows. In each round t , the auctioneer calls out the price function $p^0(t)$ and informs every bidder i of his price function $p^i(t)$. Then, based on his own bidding history and his observed information, every bidder i reports a bid or a bundle $A_i(t) \in 2^N$. Subsequently, according to reported bids and the price functions $p^{-l}(t)$, the auctioneer chooses a supply set $\gamma^l(t) \in S(p^{-l}(t))$ for every market \mathcal{M}_{-l} and adjusts all price functions through the following formulas (4.13) and (4.14).

We say a bundle S is *first-price-over-demanded by a bidder i* if bidder i is the unique bidder such that $A_i(t) = S$, $p^i(t, S) = p^{-0}(t, S)$, and $S \notin \gamma^{-l}(t)$ for some $l \in M(0) \setminus \{i\}$. A bundle S is *second-price-over-demanded* if it is demanded by at least one bidder i such that $A_i(t) = S$ and $p^i(t, S) = p^0(t, S) < p^{-0}(t, S)$, or by two or more bidders i, j ($i \neq j$) such that $A_i(t) = A_j(t) = S$ and $p^i(t, S) = p^j(t, S) = p^0(t, S) = p^{-0}(t, S)$. In general, we say a bundle S is *over-demanded* whenever it is first- or second-price-over demanded. The auctioneer adjusts the prices as follows: For every bidder $i \in M$ and every bundle $S \in 2^N$, let

$$q^0(t+1, S) = \begin{cases} p^0(t, S) + 1, & \text{if } S \text{ is 2nd-price-over-demanded,} \\ p^0(t, S), & \text{otherwise;} \end{cases}$$

$$q^i(t+1, S) = \begin{cases} p^i(t, S) + 1, & \text{if } S \text{ is 1st-price-over-demanded by bidder } i, \\ \max\{p^i(t, S), q^0(t+1, S)\}, & \text{otherwise.} \end{cases}$$

For all $i, j \in M(0)$, we define

$$p^i(t+1) = T(q^i(t+1)), \tag{4.13}$$

$$p^{-j}(t+1) = \bigvee_{i \in M_{-j}} p^i(t+1), \quad (4.14)$$

where T is the operator given by (3.8) and $a \vee b = \max\{a, b\}$ for any numbers a and b .

Formally, we have

The incentive-compatible (IC) dynamic auction mechanism

Step 1: The auctioneer announces an initial pricing function $p^0(0) : 2^N \rightarrow \mathbf{Z}_+$ so that $p^0(0, S)$ equals the reserve price $R(S)$ for every bundle $S \subseteq N$. Set $p^i(0) = p^{-i}(0) = p^0(0)$ for every $i \in M(0)$. Set $t := 0$ and go to Step 2.

Step 2: At each round $t = 0, 1, 2, \dots$, the auctioneer announces the price function $p^0(t)$ and informs every bidder $i \in M$ of his price function $p^i(t)$. Then, every bidder i reports his bid $A_i(t) \in 2^N$. For every market \mathcal{M}_{-j} ($j \in M(0)$), the auctioneer chooses a supply set $\gamma^{-j}(t) \in S(p^{-j}(t))$ so that the market \mathcal{M}_{-j} has the least over-demanded bundles. If there is no over-demanded bundle in any \mathcal{M}_{-j} , go to Step 3. Otherwise, there is at least one over-demanded bundle in some market \mathcal{M}_{-j} . Then according to formulas (4.13) and (4.14), the auctioneer adjusts the price functions $p^0(t+1), p^i(t+1)$ and $p^{-j}(t+1)$ for all $i \in M$ and $j \in M(0)$. If $p^i(t+1, S) > U^*$ for some bidder i and some bundle S , then go to Step 4. Otherwise, set $t := t+1$ and return to Step 2.

Step 3: At the last round $t = t^*$, for every $j \in M(0)$ the auctioneer chooses an allocation π^{-j} for the market \mathcal{M}_{-j} as the basic price adjustment process does (see Note 4 below in detail). According to the allocation π^{-0} of the original market \mathcal{M} , the auctioneer assigns the bundle $\pi^{-0}(0)$ to the seller and the bundle $\pi^{-0}(i)$ to bidder $i \in M$ who is asked to pay the price

$$q_i = R(\pi^{-i}(0)) + \sum_{l \in M_{-i}} p^{-i}(t^*, \pi^{-i}(l)) - R(\pi^{-0}(0)) - \sum_{l \in M_{-i}} p^{-0}(t^*, \pi^{-0}(l)) \quad (4.15)$$

Then the auction stops.

Step 4: The auctioneer assigns the whole bundle N to a bidder i with $p^i(t+1, S) > U^*$ and asks him to pay the price U^* . And all other bidders get nothing and pay nothing. The auction stops.

Note 1: Observe that in particular $p^0(0) = p^i(0) = p^{-i}(0)$, in general $p^0(t) \leq p^j(t) \leq p^{-i}(t) \leq p^{-0}(t)$ for all $i, j \in M$ ($i \neq j$), $t > 0$, and $p^i(t) - p^0(t) = p^{-0}(t) - p^{-i}(t)$, $i \in M$. Also observe that the auctioneer runs a real auction for the real market \mathcal{M} but runs a virtual auction for each virtual market \mathcal{M}_{-i} , $i \in M$ in the sense that bidders do not actually participate in the virtual markets, because they only need to submit bids for the market \mathcal{M} and the same bids are used only by the auctioneer for the virtual markets.

Note 2: It is important to point out that although every bidder $i \in M$ confronts m markets \mathcal{M}_{-j} , ($j \in M(0)$, $j \neq i$), he only needs to submit one bid $A_i(t)$ (the same bid for all m markets he faces) in each round t of the auction. This is very simple, useful and practical from the viewpoint of bidders, and is quite different from Ausubel's (2006) auction (also Sun and Yang (2008A)) where every bidder $i \in M$ confronts also m markets \mathcal{M}_{-j} , ($j \in M(0)$, $j \neq i$), but needs to submit a bid $A_i^j(t)$ for each market \mathcal{M}_{-j} (and m bids for m markets in total) in each round t of the auction.

Note 3: Observe that the bidder i 's payment q_i given by (4.15) involves only the prices and allocations of the last round t^* of the auction, is simple and easy to compute and also has an intuitive interpretation: it equals the difference between the total supposed payments of bidder i 's opponents (including the seller) in the market \mathcal{M}_{-i} without bidder i and the total supposed payments of bidder i 's opponents in the original market \mathcal{M} in the last round t^* . As will be shown below, when all bidders bid truthfully, the pair $(p^{-j}(t^*), \pi^{-j})$ is a nonlinear pricing Walrasian equilibrium in every market \mathcal{M}_{-j} and q_i equals the difference between the total equilibrium payments of bidder i 's opponents in the market \mathcal{M}_{-i} and the total equilibrium payments of bidder i 's opponents in the original market \mathcal{M} .

Note 4: For each market \mathcal{M}_{-j} , the auctioneer chooses the allocation π^{-j} as follows: (i) assign $A_i(t^*)$ to bidder $i \in M_{-j}$; (ii) assign every nonempty bundle $B \in \gamma^{-j}(t^*)$ to the seller if the bundle is not demanded by any bidder in M_{-j} and $p^{-j}(t^*, B) = R(B)$; (iii) assign every nonempty bundle $B \in \gamma^{-j}(t^*)$ to a bidder $l \in M_{-j}$ who demanded B in some round $t < t^*$ at price $p^l(t, B) = p^{-j}(t^*, B)$ or $p^{-j}(t^*, B) - 1$, if there is no bidder in M_{-j} (including bidder l) who ever demanded B at a higher price than $p^l(t, B)$, and if the bundle B is not demanded in the last round t^* by any bidder in M_{-j} and

$p^{-j}(t^*, B) > R(B)$. Clearly, in Case (iii) we have $p^l(t^*, B) = p^{-j}(t^*, B)$. For the market \mathcal{M}_{-j} , define $J(i, j) = \{l \in J \mid \gamma^{-j}(t^*, l) \text{ is nonempty and assigned to bidder } i\}$ for every bidder $i \in M_{-j}$ and $J(0, j) = \{l \in J \mid \gamma^{-j}(t^*, l) \text{ is nonempty and assigned to the seller}\}$. Then, let $\pi^{-j}(i) = \cup_{l \in J(i, j)} \gamma^{-j}(t^*, l)$ for every $i \in M_{-j}$ and $\pi^{-j}(0) = \cup_{l \in J(0, j)} \gamma^{-j}(t^*, l)$. Since $\gamma^{-j}(t^*)$ is a partition of N , it is clear that $\pi^{-j}(i) \cap \pi^{-j}(l) = \emptyset$ for $i \neq l$, and $\cup_{i \in M_{-j}} \pi^{-j}(i) = N$. This gives the allocation π^{-j} for the market \mathcal{M}_{-j} . Now we have the following lemma whose proof along with that of Theorem 4.3 is given in the Appendix.

Lemma 4.1 *For every bidder i in any market \mathcal{M}_{-j} , we have*

$$p^i(t^*, \pi^{-j}(i)) = \sum_{l \in J(i, j)} p^i(t^*, \gamma^{-j}(t^*, l)) = \sum_{l \in J(i, j)} p^{-j}(t^*, \gamma^{-j}(t^*, l)) = p^{-j}(t^*, \pi^{-j}(i)), \quad (4.16)$$

and for the seller we have

$$R(\pi^{-j}(0)) = \sum_{l \in J(0, j)} R(\gamma^{-j}(t^*, l)).$$

Moreover, the payment q_i of (4.15) can be written as

$$q_i = Re(p^{-i}(t^*)) - Re(p^{-0}(t^*)) + \sum_{l \in J(i, 0)} p^{-0}(t^*, \gamma^{-0}(t^*, l)). \quad (4.17)$$

The formula (4.17) says that the payment q_i of bidder i is equal to the seller's total revenue in market \mathcal{M}_{-i} minus her total revenue in the original market \mathcal{M} plus the prices of all bundles assigned to bidder i in the market \mathcal{M} . This formula also implies that to calculate the payment q_i of bidder i it is not necessary to consider the allocation π^{-i} of the market \mathcal{M}_{-i} .

Before studying the strategic aspect of the auction, we will demonstrate how the auction actually works via Example 1. The whole IC auction process is shown in Table 4, where the price vectors are understood as $p = (p(A), p(B), p(C), p(AB), p(AC), p(BC), p(ABC))$ with $p(\emptyset) = 0$, the seller's column indicates her supplies $\gamma = (\gamma(1), \gamma(2))$ with $\gamma(0) = \gamma(3) = \emptyset$ being ignored, whereas one column indicates each bidder's demands.

The auctioneer starts with the reserve prices $p^0(0) = (1, 1, 1, 2, 2, 2, 3)$. At the initial round $t = 0$, all bidders face the same price function $p^0(0)$, i.e., $p^i(0) = p^0(0)$ for all $i \in M$. Bidder 1 reports a bid $A_1(0) = AB \in D^1(p^i(0))$, but the auctioneer does not supply it and no other bidders report the same bundle. According to our definition, AB is not

second-price-over-demanded, but is first-price-over-demanded by bidder 1. And hence, by the price adjustment rule, $p^0(1, AB) = p^0(0, AB) = 2$ and $p^1(1, AB) = p^1(0, AB) + 1 = 3$. On the other hand, at round $t = 6$, it can be seen that the bundle AB is not first-price-over-demanded by bidder 1 because the auctioneer supplies it in all markets, but AB is second-price-over-demanded because it is demanded by bidder 3 at $p^3(6, AB) = p^0(6, AB) = 3 < p^{-0}(6, AB) = p^1(6, AB) = 6$. And so, $p^0(7, AB) = p^0(6, AB) + 1 = 3$ and $p^1(7, AB) = p^{-0}(7, AB) = p^{-0}(6, AB) = 6$. In the same time, at round $t = 6$, for the seller there are two supply sets (AB, C) and (A, BC) in the market \mathcal{M} , but she chooses the first one so that the market yields no over-demanded bundles.

The auction stops at round 9 with the allocations

$$\pi^{-0} = (\pi^{-0}(0), \pi^{-0}(1), \pi^{-0}(2), \pi^{-0}(3)) = (\emptyset, AB, C, \emptyset),$$

$$\pi^{-1} = (\pi^{-1}(0), \pi^{-1}(2), \pi^{-1}(3)) = (\emptyset, AC, B),$$

$$\pi^{-2} = (\pi^{-2}(0), \pi^{-2}(1), \pi^{-2}(3)) = (\emptyset, AB, C),$$

$$\pi^{-3} = (\pi^{-3}(0), \pi^{-3}(1), \pi^{-3}(2)) = (\emptyset, AB, C).$$

By the auction rule, bidder 1 gets AB and pays $q_1 = p^2(9, \pi^{-1}(2)) + p^3(9, \pi^{-1}(3)) - p^2(9, \pi^{-0}(2)) + p^3(9, \pi^{-0}(3)) = 6 + 2 - 2 = 6$, bidder 2 gets C and pays $q_2 = p^1(9, \pi^{-2}(1)) + p^3(9, \pi^{-2}(3)) - p^1(9, \pi^{-0}(1)) + p^3(9, \pi^{-0}(3)) = 6 + 2 - 6 = 2$, bidder 3 gets nothing and pays nothing. The pairs $(p^{-0}(9), \pi^{-0})$, $(p^{-1}(9), \pi^{-1})$, $(p^{-2}(9), \pi^{-2})$, and $(p^{-3}(9), \pi^{-3})$ are non-linear pricing Walrasian equilibria for the markets \mathcal{M} , \mathcal{M}_{-1} , \mathcal{M}_{-2} and \mathcal{M}_{-3} , respectively. The whole process of demands and supplies is shown in Table 4.

To investigate the incentive properties of this dynamic auction mechanism, we need to study an extensive-form dynamic game of incomplete information induced by the auction mechanism. In the game, bidders are the players. Prior to the start of the (auction) game, nature reveals to every player $i \in M$ only his own utility function $u^i \in \mathcal{U}$ of private information and a joint probability distribution $F(\cdot)$ from which the profile $\{u^i\}_{i \in M}$ is drawn. Let H_i^t be the part of the information (or history) of play that player i has observed just before he submits his bid at time $t \in \mathbb{Z}_+$. According to the above auction rule, a natural

Table 4: Demands and supplies at the adjusted price vectors for Example 1.

$t = 0$	$p^0(0) = (1, 1, 1, 2, 2, 2, 3)$ $p^1(0) = (1, 1, 1, 2, 2, 2, 3)$ $p^2(0) = (1, 1, 1, 2, 2, 2, 3)$ $p^3(0) = (1, 1, 1, 2, 2, 2, 3)$	$A_1(0) = AB$ $A_2(0) = AC$ $A_3(0) = BC$	$p^{-0}(0) = (1, 1, 1, 2, 2, 2, 3)$ $p^{-1}(0) = (1, 1, 1, 2, 2, 2, 3)$ $p^{-2}(0) = (1, 1, 1, 2, 2, 2, 3)$ $p^{-3}(0) = (1, 1, 1, 2, 2, 2, 3)$	$\gamma^{-0}(0) = (AB, C)$ $\gamma^{-1}(0) = (AC, B)$ $\gamma^{-2}(0) = (A, BC)$ $\gamma^{-3}(0) = (AB, C)$
$t = 1$	$p^0(1) = (1, 1, 1, 2, 2, 2, 3)$ $p^1(1) = (1, 1, 1, 3, 2, 2, 4)$ $p^2(1) = (1, 1, 1, 2, 3, 2, 4)$ $p^3(1) = (1, 1, 1, 2, 2, 3, 4)$	$A_1(1) = AB$ $A_2(1) = AC$ $A_3(1) = BC$	$p^{-0}(1) = (1, 1, 1, 3, 3, 3, 4)$ $p^{-1}(1) = (1, 1, 1, 2, 3, 3, 4)$ $p^{-2}(1) = (1, 1, 1, 3, 2, 3, 4)$ $p^{-3}(1) = (1, 1, 1, 3, 3, 2, 4)$	$\gamma^{-0}(1) = (AB, C)$ $\gamma^{-1}(1) = (AC, B)$ $\gamma^{-2}(1) = (A, BC)$ $\gamma^{-3}(1) = (AB, C)$
$t = 2$	$p^0(2) = (1, 1, 1, 2, 2, 2, 4)$ $p^1(2) = (1, 1, 1, 4, 2, 2, 5)$ $p^2(2) = (1, 1, 1, 2, 4, 2, 5)$ $p^3(2) = (1, 1, 1, 2, 2, 4, 5)$	$A_1(2) = AB$ $A_2(2) = AC$ $A_3(2) = BC$	$p^{-0}(2) = (1, 1, 1, 4, 4, 4, 5)$ $p^{-1}(2) = (1, 1, 1, 2, 4, 4, 5)$ $p^{-2}(2) = (1, 1, 1, 4, 2, 4, 5)$ $p^{-3}(2) = (1, 1, 1, 4, 4, 2, 5)$	$\gamma^{-0}(2) = (AB, C)$ $\gamma^{-1}(2) = (AC, B)$ $\gamma^{-2}(2) = (A, BC)$ $\gamma^{-3}(2) = (AB, C)$
$t = 3$	$p^0(3) = (1, 1, 1, 2, 2, 2, 5)$ $p^1(3) = (1, 1, 1, 5, 2, 2, 6)$ $p^2(3) = (1, 1, 1, 2, 5, 2, 6)$ $p^3(3) = (1, 1, 1, 2, 2, 5, 6)$	$A_1(3) = BC$ $A_2(3) = C$ $A_3(3) = AB$	$p^{-0}(3) = (1, 1, 1, 5, 5, 5, 6)$ $p^{-1}(3) = (1, 1, 1, 2, 5, 5, 6)$ $p^{-2}(3) = (1, 1, 1, 5, 2, 5, 6)$ $p^{-3}(3) = (1, 1, 1, 5, 5, 2, 6)$	$\gamma^{-0}(3) = (AB, C)$ $\gamma^{-1}(3) = (A, BC)$ $\gamma^{-2}(3) = (A, BC)$ $\gamma^{-3}(3) = (AB, C)$
$t = 4$	$p^0(4) = (1, 1, 1, 3, 2, 3, 6)$ $p^1(4) = (1, 1, 2, 5, 2, 3, 6)$ $p^2(4) = (1, 1, 2, 3, 5, 3, 6)$ $p^3(4) = (1, 1, 2, 3, 2, 5, 6)$	$A_1(4) = AB$ $A_2(4) = AC$ $A_3(4) = BC$	$p^{-0}(4) = (1, 1, 2, 5, 5, 5, 6)$ $p^{-1}(4) = (1, 1, 1, 3, 5, 5, 6)$ $p^{-2}(4) = (1, 1, 1, 5, 2, 5, 6)$ $p^{-3}(4) = (1, 1, 1, 5, 5, 3, 6)$	$\gamma^{-0}(4) = (AB, C)$ $\gamma^{-1}(4) = (AC, B)$ $\gamma^{-2}(4) = (AB, C)$ $\gamma^{-3}(4) = (AB, C)$
$t = 5$	$p^0(5) = (1, 1, 1, 3, 2, 3, 6)$ $p^1(5) = (1, 1, 1, 5, 2, 3, 7)$ $p^2(5) = (1, 1, 2, 3, 6, 3, 7)$ $p^3(5) = (1, 1, 1, 3, 2, 6, 7)$	$A_1(5) = AB$ $A_2(5) = A$ $A_3(5) = AC$	$p^{-0}(5) = (1, 1, 2, 5, 6, 6, 7)$ $p^{-1}(5) = (1, 1, 2, 3, 6, 6, 7)$ $p^{-2}(5) = (1, 1, 1, 5, 2, 6, 7)$ $p^{-3}(5) = (1, 1, 2, 5, 6, 3, 7)$	$\gamma^{-0}(5) = (AB, C)$ $\gamma^{-1}(5) = (AC, B)$ $\gamma^{-2}(5) = (A, BC)$ $\gamma^{-3}(5) = (AB, C)$
$t = 6$	$p^0(6) = (1, 1, 1, 3, 3, 3, 7)$ $p^1(6) = (1, 1, 1, 6, 3, 3, 7)$ $p^2(6) = (2, 1, 2, 3, 6, 3, 7)$ $p^3(6) = (1, 1, 1, 3, 3, 6, 7)$	$A_1(6) = AB$ $A_2(6) = \emptyset$ $A_3(6) = AB$	$p^{-0}(6) = (2, 1, 2, 6, 6, 6, 7)$ $p^{-1}(6) = (2, 1, 2, 3, 6, 6, 7)$ $p^{-2}(6) = (1, 1, 1, 6, 3, 6, 7)$ $p^{-3}(6) = (2, 1, 2, 6, 6, 3, 7)$	$\gamma^{-0}(6) = (AB, C)$ $\gamma^{-1}(6) = (AC, B)$ $\gamma^{-2}(6) = (AB, C)$ $\gamma^{-3}(6) = (AB, C)$
$t = 7$	$p^0(7) = (1, 1, 1, 4, 3, 3, 7)$ $p^1(7) = (1, 1, 1, 6, 3, 3, 7)$ $p^2(7) = (2, 1, 2, 4, 6, 3, 7)$ $p^3(7) = (1, 1, 1, 4, 3, 6, 7)$	$A_1(7) = AB$ $A_2(7) = \emptyset$ $A_3(7) = B$	$p^{-0}(7) = (2, 1, 2, 6, 6, 6, 7)$ $p^{-1}(7) = (2, 1, 2, 4, 6, 6, 7)$ $p^{-2}(7) = (1, 1, 1, 6, 3, 6, 7)$ $p^{-3}(7) = (2, 1, 2, 6, 6, 3, 7)$	$\gamma^{-0}(7) = (AB, C)$ $\gamma^{-1}(7) = (AC, B)$ $\gamma^{-2}(7) = (AB, C)$ $\gamma^{-3}(7) = (AB, C)$
$t = 8$	$p^0(8) = (1, 1, 1, 4, 3, 3, 7)$ $p^1(8) = (1, 1, 1, 6, 3, 3, 7)$ $p^2(8) = (2, 1, 2, 4, 6, 3, 7)$ $p^3(8) = (1, 2, 1, 4, 3, 6, 7)$	$A_1(8) = AB$ $A_2(8) = \emptyset$ $A_3(8) = C$	$p^{-0}(8) = (2, 2, 2, 6, 6, 6, 7)$ $p^{-1}(8) = (2, 2, 2, 4, 6, 6, 7)$ $p^{-2}(8) = (1, 2, 1, 6, 3, 6, 7)$ $p^{-3}(8) = (2, 1, 2, 6, 6, 3, 7)$	$\gamma^{-0}(8) = (AB, C)$ $\gamma^{-1}(8) = (AC, B)$ $\gamma^{-2}(8) = (AB, C)$ $\gamma^{-3}(8) = (AB, C)$
$t = 9$	$p^0(9) = (1, 1, 2, 4, 3, 3, 7)$ $p^1(9) = (1, 1, 2, 6, 3, 3, 7)$ $p^2(9) = (2, 1, 2, 4, 6, 3, 7)$ $p^3(9) = (1, 2, 2, 4, 3, 6, 7)$	$A_1(9) = AB$ $A_2(9) = \emptyset$ $A_3(9) = \emptyset$	$p^{-0}(9) = (2, 2, 2, 6, 6, 6, 7)$ $p^{-1}(9) = (2, 2, 2, 4, 6, 6, 7)$ $p^{-2}(9) = (1, 2, 2, 6, 3, 6, 7)$ $p^{-3}(9) = (2, 1, 2, 6, 6, 3, 7)$	$\pi^{-0} = (\emptyset, AB, C, \emptyset)$ $\pi^{-1} = (\emptyset, AC, B)$ $\pi^{-2} = (\emptyset, AB, C)$ $\pi^{-3} = (\emptyset, AB, C)$

and sensible specification is that H_i^t should comprise all observable price functions and

his own bids, i.e.,

$$H_i^t = \{p^i(t), p^i(s), p^0(t), p^0(s), A_i(s) \mid 0 \leq s < t\}.$$

After any history H_i^t and at any time $t \in \mathbf{Z}_+$, each player i updates his posterior beliefs $\mu_i(\cdot \mid t, H_i^t, u^i)$ over opponents' utility functions. We stress that even after the auction is finished, player i may not know his opponents' utility functions precisely.

A (dynamic) *strategy* σ_i of player i is a function $\{(t, H_i^t, u^i) \mid t \in \mathbf{Z}_+, u^i \in \mathcal{U}\} \rightarrow 2^N$, which tells him to bid $\sigma_i(t, H_i^t, u^i) = A_i(t) \in 2^N$ at each time $t \in \mathbf{Z}_+$ when he observes H_i^t . Let Σ_i denote player i 's strategy space of all such strategies σ_i . We say that a strategy σ_i is a *sincere bidding strategy* for player i if he always bids sincerely with respect to his true utility function u^i for any $t \in \mathbf{Z}_+$ and any price function $p^i(t)$. Observe that sincere bidding strategy may not be unique, and the strategy space Σ_i of player i contains sincere bidding strategies and also various other strategies.

Given the auction rules, the outcome of this auction game depends entirely upon the realization of utility functions and the strategies the bidders take. When every bidder $i \in M$ takes a strategy σ_i and the IC dynamic auction terminates at round $t = t^*$, then bidder i receives bundle $\pi^{-0}(i)$ and pays q_i given by (4.15). When every bidder $i \in M$ takes a strategy σ_i and the dynamic auction stops in Step 4, let i^* denote the player i^* who receives the whole bundle N . Then, we have $\pi^{-0}(i^*) = N$, $q_{i^*} = U^*$, $\pi^{-0}(i) = \emptyset$ and $q_i = 0$ for all $i \in M_{-i^*}$. So in this auction game, every player i 's payoff function $W_i(\cdot, \cdot)$ is given by

$$W_i(\{\sigma_l\}_{l \in M}, \{u^l\}_{l \in M}) = u^i(\pi^{-0}(i)) - q_i. \quad (4.18)$$

For auction games of incomplete information, the ex post Nash equilibrium was used by Crémer and McLean (1985) for a sealed-bid auction (see also Krishna (2002)) and the ex post perfect equilibrium by Ausubel (2006), Sun and Yang (2008A) for a dynamic auction. Stronger than Bayesian equilibrium or perfect Bayesian equilibrium, these notions of equilibrium have a number of additional desirable properties, i.e., they are not only robust against any regret but also independent of any probability distribution.

We shall use the notion of ex post Nash equilibrium. An m -tuple $\{\sigma_i\}_{i \in M}$ is an *ex post Nash equilibrium*, if for any realization $\{u^l\}_{l \in M}$ of profile of utility functions of private

information, and for any strategy $\sigma_i' \in \Sigma_i$ of any player $i \in M$, we have

$$W_i(\sigma_i, \{\sigma_l\}_{l \in M_{-i}}, \{u^l\}_{l \in M}) \geq W_i(\sigma_i', \{\sigma_l\}_{l \in M_{-i}}, \{u^l\}_{l \in M}).$$

In other words, an m -tuple $\{\sigma_i\}_{i \in M}$ is an ex post Nash equilibrium, whenever for any realization $\{u^l\}_{l \in M}$ of profile of utility functions of private information, the strategy σ_i of every player $i \in M$ constitutes his best response against the strategies $\{\sigma_l\}_{l \in M_{-i}}$ of all his opponents even if the realization $\{u^l\}_{l \in M}$ becomes common knowledge.

We say an auction mechanism is *ex post individually rational*, whenever for every bidder, no matter how his opponents bid, as long as he bids sincerely he will never obtain a negative payoff. More formally, an auction mechanism is ex post individually rational, whenever for any realization $\{u^l\}_{l \in M}$ of profile of utility functions of private information, any sincere bidding strategy σ_i of any bidder i , and any bidding strategies $\{\sigma_l'\}_{l \in M_{-i}} \in \{\Sigma_l\}_{l \in M_{-i}}$ of bidder i 's opponents, it holds that

$$W_i(\sigma_i, \{\sigma_l'\}_{l \in M_{-i}}, \{u^l\}_{l \in M}) \geq 0.$$

Now we briefly recall the (sealed-bid) VCG mechanism. In this mechanism every bidder $i \in M$ is required to report his utility function u^i to the auctioneer. Then she computes an efficient allocation π with respect to all bidders' reported u^i and assigns bundle $\pi(i)$ to bidder i and charges him a payment of $q_i^* = u^i(\pi(i)) - V(N) + V_{-i}(N)$, where $V(N)$ and $V_{-i}(N)$ are the market values of the markets \mathcal{M} and \mathcal{M}_{-i} based on u^i ($i \in M$) and her own reserve price function R . Bidder i 's VCG payoff equals $V(N) - V_{-i}(N)$. We will call the payment q_i^* *the generalized VCG payment* of bidder i because the payment here also involves the reserve price function R of the seller whereas in the VCG mechanism the reserve price is typically assumed to be zero.

The following theorem shows a useful property of the IC dynamic auction that although each bidder i submits the same bid for all markets according to his price function $p^i(t)$ rather than every market price function $p^{-j}(t)$, the auction terminates with a nonlinear pricing Walrasian equilibrium in every market.

Theorem 4.2 *Suppose that Assumptions (A1)–(A4) hold for the auction model. If every bidder bids sincerely, the IC dynamic auction must stop in Step 3 at some round t^* and*

for every $j \in M(0)$, the pair $(p^{-j}(t^*), \pi^{-j})$ is a nonlinear pricing Walrasian equilibrium for the market \mathcal{M}_{-j} .

Proof: Consider any market \mathcal{M}_{-j} , $j \in M(0)$. Then $p^{-j}(t^*)$ is the price function of the auctioneer for the market \mathcal{M}_{-j} and π^{-j} is the allocation chosen by the auctioneer for the market \mathcal{M}_{-j} in the last round t^* . When every bidder $i \in M_{-j}$ bids sincerely, the bundle $A_i(t^*)$ reported by bidder i in the last round t^* must be one of his true demand bundles, i.e., $A_i(t^*) \in D^i(p^i(t^*))$. Recall that $p^i(t^*, A_i(t^*)) = p^{-j}(t^*, A_i(t^*))$ for every $i \in M_{-j}$. Thus, it follows from $p^i(t^*) \leq p^{-j}(t^*)$ that $A_i(t^*) \in D^i(p^{-j}(t^*))$ and

$$V^i(p^i(t^*)) = u^i(A_i(t^*)) - p^i(t^*, A_i(t^*)) = u^i(A_i(t^*)) - p^{-j}(t^*, A_i(t^*)) = V^i(p^{-j}(t^*)).$$

Furthermore, by the assignment rule, for any bidder $i \in M_{-j}$ and any bundle $\gamma^{-j}(t^*, l)$ which is forcibly assigned to him, i.e., $l \in J(i, j)$ and $\gamma^{-j}(t^*, l) \neq A_i(t^*)$, when he bids sincerely at all rounds we have $u^i(\gamma^{-j}(t^*, l)) - p^{-j}(t^*, \gamma^{-j}(t^*, l)) \geq 1$ or $u^i(\gamma^{-j}(t^*, l)) - (p^{-j}(t^*, \gamma^{-j}(t^*, l)) - 1) \geq 1$, and so $u^i(\gamma^{-j}(t^*, l)) - p^{-j}(t^*, \gamma^{-j}(t^*, l)) \geq 0$.

Note that when $A_i(t^*) \neq \emptyset$, it is always assigned to bidder i , i.e., there is some $l \in J(i, j)$ such $\gamma^{-j}(t^*, l) = A_i(t^*)$. Now it follows from that Assumption (A3) and (4.16) that

$$\begin{aligned} & u^i(\pi^{-j}(i)) - p^{-j}(t^*, \pi^{-j}(i)) \\ &= u^i(\pi^{-j}(i)) - \sum_{l \in J(i, j)} p^{-j}(t^*, \gamma^{-j}(t^*, l)) \\ &\geq \sum_{l \in J(i, j)} u^i(\gamma^{-j}(t^*, l)) - \sum_{l \in J(i, j)} p^{-j}(t^*, \gamma^{-j}(t^*, l)) \\ &= \sum_{l \in J(i, j)} (u^i(\gamma^{-j}(t^*, l)) - p^{-j}(t^*, \gamma^{-j}(t^*, l))) \\ &\geq u^i(A_i(t^*)) - p^{-j}(t^*, A_i(t^*)) \\ &= V^i(p^{-j}(t^*)). \end{aligned} \tag{4.19}$$

Notice that even if $A_i(t^*) = \emptyset$, the above inequality is still true. This implies that $\pi^{-j}(i) \in D^i(p^{-j}(t^*))$ for every $i \in M_{-j}$. Moreover, by (4.17) we know that the allocation $\pi^{-j} \in S(p^{-j}(t^*))$. So we can conclude that $(p^{-j}(t^*), \pi^{-j})$ is a nonlinear pricing Walrasian equilibrium of the market \mathcal{M}_{-j} . \square

Now we can present the major result of this section concerning the auction.

Theorem 4.3 *Suppose that the market \mathcal{M} satisfies Assumptions (A1)–(A4).*

(i) *The IC dynamic auction is ex post individually rational.*

(ii) When every bidder bids sincerely, the IC dynamic auction yields a Vickrey-Clarke-Groves outcome for the market \mathcal{M} in a finite number of rounds. Moreover, it holds $q_i \geq R(\pi^{-0}(i))$ for every bidder $i \in M$.

(iii) Sincere bidding is an ex post Nash equilibrium in the game induced by the IC dynamic auction.

Analogously, we can obtain the following similar results for the IC dynamic auction if every bidder uses quasi-sincere bidding strategies.

Theorem 4.4 *Suppose that Assumptions (A1)–(A4) hold for the auction model. If every bidder bids quasi-sincerely, the IC dynamic auction must stop in Step 3 at some round t^* and for every $j \in M(0)$, the pair $(p^{-j}(t^*), \pi^{-j})$ is a nonlinear pricing Walrasian equilibrium for the market \mathcal{M}_{-j} .*

Theorem 4.5 *Suppose that the market \mathcal{M} satisfies Assumptions (A1)–(A4).*

(i) *The IC dynamic auction is strongly ex post individually rational.*

(ii) *When every bidder bids quasi-sincerely, the IC dynamic auction yields a Vickrey-Clarke-Groves outcome for the market \mathcal{M} in a finite number of rounds. Moreover, it holds $q_i \geq R(\pi^{-0}(i))$ for every bidder $i \in M$.*

(iii) *Quasi-sincere bidding is an ex post Nash equilibrium in the game induced by the IC dynamic auction.*

The current dynamic auction yields the same outcome as that of the VCG mechanism, but offers several advantages over the VCG auction: First, it utilizes information from every buyer efficiently and judiciously in that it only requires him to report his demand sets on a number of price vectors, whereas the VCG mechanism is sealed-bid and requires every buyer to report his entire values. So the current auction possesses the important privacy-preserving property, which is very significant in practice, because in reality businessmen generally do not like to reveal their values even if truth-telling may be theoretically the best strategy; see e.g., Rothkopf (2007). Second, the current auction gives a simple and transparent way of computing efficient allocations, equilibrium prices and VCG payments using observable information, whereas the VCG auction tells only a way of computing VCG payments assuming that all buyers' values and efficient allocations are already given.

5 Applications

Situations involving complementarities abound. Economies of scale generate complementarities. To name but a few, different segments of a railroad network (see Brewer and Plott (1996)) are typically complements, and so are different parts of a machine, players of a soccer team and books in several volumes; see also Rassenti, Smith and Bulfin (1982), Ostrovsky (2008) and Sun and Yang (2006A).

As a simple application, we discuss a multi-sided matching equilibrium model. Suppose that a seller wishes to sell three complementary sets A , B and C of indivisible goods to a number of buyers $M = \{1, 2, \dots, m\}$. (The analysis here applies to four or more complementary sets of goods as well). For instance, one may think of A as main processors (CPUs), of B as displays, and of C as keyboards. Let $A = \{a_1, a_2, \dots, a_r\}$, $B = \{b_1, b_2, \dots, b_s\}$, and $C = \{c_1, c_2, \dots, c_t\}$. Without loss of generality, we may assume that the seller values every bundle of goods at zero. Each buyer has a sufficient amount of money and can buy as many goods as he wishes. The value of buyer $i \in M$ on the combination of a CPU $a \in A$, a display $b \in B$ and a keyboard $c \in C$ is $v_i(a, b, c)$, which is assumed to be a nonnegative integer value. If buyer $i \in M$ uses a set S of CPUs, displays and keyboards, the value $u^i(S)$ of these CPUs, displays and keyboards to the buyer is completely determined by the triplewise combinations of CPU, display and keyboard that the goods in S can generate, and can be explicitly formulated as

$$u^i(S) = \max\{0, v_i(a_{i_1}, b_{j_1}, c_{k_1}) + v_i(a_{i_2}, b_{j_2}, c_{k_2}) + \dots + v_i(a_{i_l}, b_{j_l}, c_{k_l})\}$$

with the maximum to be taken over all sets $\{(a_{i_1}, b_{j_1}, c_{k_1}), (a_{i_2}, b_{j_2}, c_{k_2}), \dots, (a_{i_l}, b_{j_l}, c_{k_l})\}$ of l distinct CPU-display-keyboard matchings in S . The valuation formula u^i for every buyer $i \in M$ can be seen as an extension of the worker-machine assignment valuations derived by Shapley (1962). Hatfield and Milgrom (2005) give a different extension of Shapley's assignment model.

So in this model, every buyer faces internally an optimal three-sided (CPU-display-keyboard) assignment problem and the whole economy faces a more complex optimal four-sided (buyer-CPUs-displays-keyboards) assignment problem. The following theorem shows that for this model there is a nonlinear system of competitive prices through which all

CPUs, displays and keyboards can be efficiently allocated to the buyers. But the model may not have a standard Walrasian equilibrium.

Theorem 5.1 *The value function u^i of every buyer $i \in M$ satisfies Assumptions (A1)–(A3) and thus the model has a nonlinear pricing Walrasian equilibrium. But the model may not have a standard Walrasian equilibrium.*

Proof: It is obvious that the value function u^i of every buyer $i \in M$ satisfies Assumptions (A1) and (A2). Now let us show that u^i satisfies Assumption (A3). Let $N = A \cup B \cup C$. For any $S, S' \in 2^N$ with $S \cap S' = \emptyset$, by the definition of bidder i 's utility function, we have distinct CPU-display-keyboard matchings $\{(a_{i_1}, b_{j_1}, c_{k_1}), \dots, (a_{i_l}, b_{j_l}, c_{k_l})\}$ in S such that

$$u^i(S) = v_i(a_{i_1}, b_{j_1}, c_{k_1}) + \dots + v_i(a_{i_l}, b_{j_l}, c_{k_l}),$$

and distinct CPU-display-keyboard matchings $\{(a_{i'_1}, b_{j'_1}, c_{k'_1}), \dots, (a_{i'_l}, b_{j'_l}, c_{k'_l})\}$ in S' such that

$$u^i(S') = v_i(a_{i'_1}, b_{j'_1}, c_{k'_1}) + \dots + v_i(a_{i'_l}, b_{j'_l}, c_{k'_l}).$$

Observe that $\{(a_{i_1}, b_{j_1}, c_{k_1}), \dots, (a_{i_l}, b_{j_l}, c_{k_l}), (a_{i'_1}, b_{j'_1}, c_{k'_1}), \dots, (a_{i'_l}, b_{j'_l}, c_{k'_l})\}$ are also distinct CPU-display-keyboard matchings in $S \cup S'$. Thus, by the definition of u^i , we obtain that

$$\begin{aligned} u^i(S \cup S') &\geq v_i(a_{i_1}, b_{j_1}, c_{k_1}) + \dots + v_i(a_{i_l}, b_{j_l}, c_{k_l}) \\ &\quad + v_i(a_{i'_1}, b_{j'_1}, c_{k'_1}) + \dots + v_i(a_{i'_l}, b_{j'_l}, c_{k'_l}) \\ &= u^i(S) + u^i(S'). \end{aligned}$$

This shows that u^i satisfies Assumption (A3). Furthermore, by Theorem 2.4, we see that the model has a nonlinear pricing Walrasian equilibrium.

We will show the last statement by an example. Consider an economy in which a seller wants to sell two CPUs $\{a_1, a_2\}$, two displays $\{b_1, b_2\}$ and two keyboards $\{c_1, c_2\}$ to two buyers $\{1, 2\}$. The values of the buyers on CPU-display-keyboard matchings are given by $v_1(a_2, b, c) = v_2(a_1, b, c) = 1$ for any $b \in \{b_1, b_2\}$ and $c \in \{c_1, c_2\}$ and the rest are given in Table 4. In this economy there is only one efficient allocation, namely, buyer 1 is assigned the bundle $\{a_1, b_1, c_1\}$ and buyer 2 the bundle $\{a_2, b_2, c_2\}$. We will prove that this

allocation cannot be supported by a price vector $(p_{a_1}, p_{a_2}, p_{b_1}, p_{b_2}, p_{c_1}, p_{c_2})$. Suppose to the contrary that such a price vector does exist. Then, we must have the following system of inequalities:

$$12 - p_{a_1} - p_{b_1} - p_{c_1} \geq 20 - p_{a_1} - p_{b_2} - p_{c_2}$$

$$12 - p_{a_2} - p_{b_2} - p_{c_2} \geq 11 - p_{a_2} - p_{b_2} - p_{c_1}$$

$$12 - p_{a_2} - p_{b_2} - p_{c_2} \geq 11 - p_{a_2} - p_{b_1} - p_{c_2}$$

where the 1st inequality holds for buyer 1 and other two hold for buyer 2. It follows that $p_{b_2} - p_{b_1} \geq 8 + p_{c_1} - p_{c_2}$, $p_{c_1} - p_{c_2} \geq -1$, and $1 \geq p_{b_2} - p_{b_1}$. These inequalities imply $1 \geq p_{b_2} - p_{b_1} \geq 4 + p_{c_1} - p_{c_2} \geq 7$, which is impossible. Thus the economy has no standard Walrasian equilibrium.

Table 4: The values of the buyer on matchings.

$v_1(a_1, b, c)$	c_1	c_2	$v_2(a_2, b, c)$	c_1	c_2
b_1	<u>12</u>	11	b_1	2	11
b_2	11	20	b_2	11	<u>12</u>

□

Moreover, it is not difficult to establish the following characterization of the equilibrium pricing function for this model due to its specific structure.

Theorem 5.2 *The equilibrium pricing function p^* of the multi-sided matching equilibrium model can be specified by $p^*(a, b, c)$ for any $a \in A, b \in B$ and $c \in C$, namely, for any $S \subset A \cup B \cup C$,*

$$p^*(S) = \max\{0, p^*(a_{i_1}, b_{j_1}, c_{k_1}) + p^*(a_{i_2}, b_{j_2}, c_{k_2}) + \cdots + p^*(a_{i_l}, b_{j_l}, c_{k_l})\}$$

with the maximum to be taken over all sets $\{(a_{i_1}, b_{j_1}, c_{k_1}), (a_{i_2}, b_{j_2}, c_{k_2}), \cdots, (a_{i_l}, b_{j_l}, c_{k_l})\}$ of l distinct three-sided matchings in S .

Finally, it is worthwhile to note that there is a marked difference between three (or less)-sided matching model and four (or more)-sided matching model. While both two-sided matching model of Shapley and Shubik (1972) and three-sided matching model of Sun

and Yang (2006A) have standard Walrasian equilibria, the four (or more)-sided matching model as considered here need not have a standard Walrasian equilibrium but does have a nonlinear pricing Walrasian equilibrium.

6 Concluding Remarks

We have proposed an ascending auction using a uniform pricing scheme that yields an efficient allocation of items and its supporting prices within a finite number of steps, provided that the values of every bidder over items are integral and superadditive. Because the auction rules are simple, intuitive and transparent, and the sufficient conditions are quite general, it will not be difficult to put the auction on laboratory tests or to use in practice.

In the present paper, as in most of the auctions literature we concentrated on the private value setting and the auction we proposed provides a useful and necessary basis for the study of efficient auction design in the more complex environments. We believe it is important but also significantly more difficult to investigate if and how the current auction can be extended to the interdependent value setting. For auctions with interdependent values, the interested reader may consult with Milgrom and Weber (1982), Crémer and McLean (1985), Maskin (1992), Dasgupta and Maskin (2000), Maskin and Riley (2000), Jehiel and Moldovanu (2001), Krishna (2003), Ausubel (2004), Perry and Reny (2002, 2005), Bergemann and Morris (2007) among others. But most of these auctions are sealed-bid auctions. Among a few exceptions, Ausubel (2004), Perry and Reny (2005) propose ascending auctions for selling multiple identical goods.

APPENDIX

Proof of Lemma 2.2 Because (p^*, π^*) is a nonlinear pricing Walrasian equilibrium, for any bidder $i \in M$ and any allocation $\rho \in \mathcal{A}$, it holds

$$u^i(\pi^*(i)) - p^*(\pi^*(i)) \geq u^i(\rho(i)) - p^*(\rho(i)).$$

It follows that

$$\begin{aligned} & R(\pi^*(0)) + \sum_{i=1}^m u^i(\pi^*(i)) - (R(\rho(0)) + \sum_{i=1}^m u^i(\rho(i))) \\ & \geq R(\pi^*(0)) + \sum_{i=1}^m p^*(\pi^*(i)) - (R(\rho(0)) + \sum_{i=1}^m p^*(\rho(i))). \end{aligned}$$

For the seller we have

$$R(\pi^*(0)) + \sum_{i=1}^m p^*(\pi^*(i)) - (R(\rho(0)) + \sum_{i=1}^m p^*(\rho(i))) \geq 0.$$

Consequently, we have

$$R(\pi^*(0)) + \sum_{i=1}^m u^i(\pi^*(i)) \geq R(\rho(0)) + \sum_{i=1}^m u^i(\rho(i)),$$

for all $\rho \in \mathcal{A}$. This implies that π^* is efficient.

Suppose that ρ is an efficient allocation. Then we have $V(N) = R(\rho(0)) + \sum_{i \in M} u^i(\rho(i))$. Because (p^*, π^*) is a nonlinear pricing Walrasian equilibrium, π^* is an efficient allocation and thus $V(N) = R(\pi^*(0)) + \sum_{i \in M} u^i(\pi^*(i))$. Furthermore it holds

$$\begin{aligned} V^i(p^*) & \geq u^i(\rho(i)) - p^*(\rho(i)), & \text{for all } i \in M \\ \max_{\gamma \in \mathcal{B}} (R(\gamma(0)) + \sum_{j \in J_{-0}} p^*(\gamma(j))) & = R(\pi^*(0)) + \sum_{i \in M} p^*(\pi^*(i)) \\ & \geq R(\rho(0)) + \sum_{i \in M} p^*(\rho(i)). \end{aligned}$$

If one of the above inequalities were strict, we would have

$$\begin{aligned} V(N) & = R(\pi^*(0)) + \sum_{i \in M} u^i(\pi^*(i)) \\ & = R(\pi^*(0)) + \sum_{i \in M} (u^i(\pi^*(i)) - p^*(\pi^*(i))) + \sum_{i \in M} p^*(\pi^*(i)) \\ & = \sum_{i \in M} V^i(p^*) + \max_{\gamma \in \mathcal{B}} (R(\gamma(0)) + \sum_{j \in J_{-0}} p^*(\gamma(j))) \\ & > \sum_{i \in M} (u^i(\rho(i)) - p^*(\rho(i))) + R(\rho(0)) + \sum_{i \in M} p^*(\rho(i)) \\ & = R(\rho(0)) + \sum_{i \in M} u^i(\rho(i)) \\ & = V(N), \end{aligned}$$

yielding a contradiction. We therefore have

$$\begin{aligned} V^i(p^*) & = u^i(\rho(i)) - p^*(\rho(i)), & \text{for all } i \in M, \\ \max_{\gamma \in \mathcal{B}} (R(\gamma(0)) + \sum_{j \in J_{-0}} p^*(\gamma(j))) & = R(\rho(0)) + \sum_{i \in M} p^*(\rho(i)). \end{aligned}$$

This shows that ρ is also supported by p^* . We are done. \square

Proof of Lemma 2.3: Assume that p is a nonlinear equilibrium pricing function. Then by Lemma 2.2 (p, π) is a nonlinear pricing Walrasian equilibrium. That is, $\pi(i) \in D^i(p)$

for all $i \in M$ and $\pi \in S(p)$. For every bundle $A \in 2^N \setminus \{\emptyset\}$, there is a set I of agents such that $A = \cup_{i \in I} \pi(i)$. For such bundle A , let $q(A) = \sum_{i \in I} p(\pi(i))$. This defines a linear pricing function q . We will show that q is in fact an equilibrium pricing function. For each nonempty bundle A written as $A = \cup_{i \in I} \pi(i)$, choose an agent $i^* \in I$ and define a partition ρ by

$$\rho(i) = \begin{cases} A & \text{if } i = i^* \\ \emptyset & \text{if } i \in I \setminus \{i^*\} \\ \pi(i) & \text{if } i \in M_0 \setminus I. \end{cases}$$

Recall that $R(\pi(0)) + \sum_{i \in M} p(\pi(i)) \geq R(\rho(0)) + \sum_{i \in M} p(\rho(i))$. We have that $\sum_{i \in I} p(\pi(i)) \geq p(A)$. That is, $q(A) \geq p(A)$ for every bundle $A \in 2^N$. It follows that $u^i(\pi(i)) - q(\pi(i)) = u^i(\pi(i)) - p(\pi(i)) \geq u^i(A) - p(A) \geq u^i(A) - q(A)$ for every $i \in M$ and $A \in 2^N$. So, $\pi(i) \in D^i(q)$ for all $i \in M$. Next, because q is a linear pricing function, we have $R(\pi(0)) + \sum_{i \in M} q(\pi(i)) = R(\rho(0)) + \sum_{i \in M} q(\rho(i))$ for any allocation $\rho \in \mathcal{A}$. Consequently, (q, π) is a linear pricing (i.e., standard) Walrasian equilibrium. \square

Proof of Lemma 4.1: Observe that in the market \mathcal{M}_{-j} every nonempty $A_i(t^*)$ is not over-demanded, so it cannot hold that $p^i(t^*, A_i(t^*)) = p^0(t^*, A_i(t^*)) < p^{-0}(t^*, A_i(t^*))$. Hence, $p^i(t^*, A_i(t^*)) = p^{-0}(t^*, A_i(t^*)) = p^{-j}(t^*, A_i(t^*))$ for all $j \in M(0) \setminus \{i\}$. By the assignment rule in Note 4 we also have $p^i(t^*, \gamma^{-j}(t^*, l)) = p^{-j}(t^*, \gamma^{-j}(t^*, l))$ for all $l \in J(i, j)$. Recall that every $p^j(t^*)$ is a superadditive price function. Hence, for every bidder i in \mathcal{M}_{-j} , it holds that

$$p^i(t^*, \pi^{-j}(i)) \geq \sum_{l \in J(i, j)} p^i(t^*, \gamma^{-j}(t^*, l)) = \sum_{l \in J(i, j)} p^{-j}(t^*, \gamma^{-j}(t^*, l)).$$

On the other hand, for the seller it holds that

$$\sum_{l \in J(i, j)} p^{-j}(t^*, \gamma^{-j}(t^*, l)) \geq p^{-j}(t^*, \pi^{-j}(i)) \geq p^i(t^*, \pi^{-j}(i)),$$

where the second inequality is due to the fact that $p^{-j}(t^*) \geq p^i(t^*)$ for every $i \in M_{-j}$.

Therefore, for every bidder i in \mathcal{M}_{-j} we have

$$\begin{aligned} p^i(t^*, \pi^{-j}(i)) &= \sum_{l \in J(i, j)} p^i(t^*, \gamma^{-j}(t^*, l)) \\ &= \sum_{l \in J(i, j)} p^{-j}(t^*, \gamma^{-j}(t^*, l)) = p^{-j}(t^*, \pi^{-j}(i)). \end{aligned}$$

Similarly, for the seller we can show that

$$R(\pi^{-j}(0)) = \sum_{l \in J(0,j)} R(\gamma^{-j}(t^*, l)).$$

Consequently, the payment formula (4.15) can be written as

$$\begin{aligned} q_i &= R(\gamma^{-i}(t^*, 0)) + \sum_{l \in J_{-0}} p^{-i}(t^*, \gamma^{-i}(t^*, l)) - R(\gamma^{-0}(t^*, 0)) \\ &\quad - \sum_{l \in J_{-0} \setminus J(i,0)} p^{-0}(t^*, \pi^{-0}(t^*, l)) \\ &= Re(p^{-i}(t^*)) - Re(p^{-0}(t^*)) + \sum_{l \in J(i,0)} p^{-0}(t^*, \gamma^{-0}(t^*, l)). \end{aligned}$$

□

Proof of Theorem 4.3: We first prove (i). Suppose that bidder i bids sincerely, but his opponents use some other strategies $\{\hat{\sigma}_l\}_{l \in M_{-i}}$. If the auction stops in Step 4, it is obvious that bidder i gets nothing and pays nothing, and so $W_i(\sigma_i, \{\hat{\sigma}_l\}_{l \in M_{-i}}, \{u^l\}_{l \in M}) = 0$. Now assume that the auction stops in Step 3 at round $t = \hat{t}$, and finds an allocation $\hat{\pi}^{-j}$ in each market \mathcal{M}_{-j} respectively. Let $\hat{p}^i(t)$, $\hat{p}^{-j}(t)$, and $\hat{\gamma}^{-l}(t)$ be the price functions and the supply sets generated by the auction. First, note that $\hat{p}^{-0}(\hat{t}) \geq \hat{p}^{-i}(\hat{t})$. Hence, the seller's revenue function satisfies

$$Re(\hat{p}^{-0}(\hat{t})) \geq Re(\hat{p}^{-i}(\hat{t})).$$

Next, let $\hat{J}(i, 0) = \{l \in J \mid \hat{\gamma}^{-0}(\hat{t}, l) \in \hat{\pi}^{-0}(i)\}$. Then, when bidder i bids sincerely, for every bundle $\hat{\gamma}^{-0}(\hat{t}, l)$ which is assigned to him (i.e., $l \in \hat{J}(i, 0)$) it satisfies

$$u^i(\hat{\gamma}^{-0}(\hat{t}, l)) - \hat{p}^{-0}(\hat{t}, \hat{\gamma}^{-0}(\hat{t}, l)) = u^i(\hat{\gamma}^{-0}(\hat{t}, l)) - \hat{p}^i(\hat{t}, \hat{\gamma}^{-0}(\hat{t}, l)) \geq 0.$$

As a result, by (4.17) and Assumption (A3), we see that for every realization $\{u^l\}_{l \in M}$

$$\begin{aligned} &W_i(\sigma_i, \{\hat{\sigma}_l\}_{l \in M_{-i}}, \{u^l\}_{l \in M}) \\ &= u^i(\hat{\pi}^{-0}(i)) - Re(\hat{p}^{-i}(\hat{t})) + Re(\hat{p}^{-0}(\hat{t})) - \sum_{l \in \hat{J}(i,0)} \hat{p}^{-0}(\hat{t}, \hat{\gamma}^{-0}(\hat{t}, l)) \\ &\geq u^i(\hat{\pi}^{-0}(i)) - \sum_{l \in \hat{J}(i,0)} \hat{p}^{-0}(\hat{t}, \hat{\gamma}^{-0}(\hat{t}, l)) \\ &\geq \sum_{l \in \hat{J}(i,0)} (u^i(\hat{\gamma}^{-0}(\hat{t}, l)) - \hat{p}^{-0}(\hat{t}, \hat{\gamma}^{-0}(\hat{t}, l))) \\ &\geq 0. \end{aligned}$$

Next we prove (ii). From Theorem 4.2 we know that when all bidders bid sincerely, the auction must terminate at some round $t = t^*$ in Step 3. Now it is sufficient to prove that q_i coincides with the generalized VCG payment $q_i^* = u^i(\pi^{-0}(i)) - V(N) + V_{-i}(N)$.

By Theorem 4.2, the pair $(p^{-i}(t^*), \pi^{-i})$ is a Walrasian equilibrium for the market \mathcal{M}_{-i} . And so each allocation π^{-i} is efficient for \mathcal{M}_{-i} . In addition, at the last round t^* , every bidder i in every market \mathcal{M}_{-j} , ($j \in M(0) \setminus \{i\}$) faces the same price function $p^i(t^*)$. Now it is easy but important to see that even if bidder i may be assigned with different bundles $\pi^{-j}(i)$ in different markets \mathcal{M}_{-j} , he obtains the same indirect utility $V^i(p^i(t^*))$. Using this observation we have:

$$\begin{aligned}
q_i^* &= u^i(\pi^{-0}(i)) - V(N) + V_{-i}(N) \\
&= u^i(\pi^{-0}(i)) - \left(R(\pi^{-0}(0)) + \sum_{l \in M} u^l(\pi^{-0}(l)) \right) \\
&\quad + \left(R(\pi^{-i}(0)) + \sum_{l \in M_{-i}} u^l(\pi^{-i}(l)) \right) \\
&= \left(R(\pi^{-i}(0)) + \sum_{l \in M_{-i}} u^l(\pi^{-i}(l)) \right) - \left(R(\pi^{-0}(0)) + \sum_{l \in M_{-i}} u^l(\pi^{-0}(l)) \right) \\
&= \left(R(\pi^{-i}(0)) + \sum_{l \in M_{-i}} (V^l(p^l(t^*)) + p^l(t^*, \pi^{-i}(l))) \right) \\
&\quad - \left(R(\pi^{-0}(0)) + \sum_{l \in M_{-i}} (V^l(p^l(t^*)) + p^l(t^*, \pi^{-0}(l))) \right) \\
&= R(\pi^{-i}(0)) + \sum_{l \in M_{-i}} p^l(t^*, \pi^{-i}(l)) - R(\pi^{-0}(0)) - \sum_{l \in M_{-i}} p^l(t^*, \pi^{-0}(l)) \\
&= R(\pi^{-i}(0)) + \sum_{l \in M_{-i}} p^{-i}(t^*, \pi^{-i}(l)) - R(\pi^{-0}(0)) - \sum_{l \in M_{-i}} p^{-0}(t^*, \pi^{-0}(l)) \\
&= q_i.
\end{aligned}$$

Thus, the payoff W_i of every bidder i is equal to his VCG payoff, i.e.,

$$W_i(\sigma_i, \{\sigma_l\}_{l \in M_{-i}}, \{u^l\}_{l \in M}) = u^i(\pi^{-0}(i)) - q_i^* = V(N) - V_{-i}(N) \geq 0.$$

Next, note that $V_{-i}(N) \geq R(\pi^{-0}(0) \cup \pi^{-0}(i)) + \sum_{l \in M_{-i}} u^l(\pi^{-0}(l))$, since $V_{-i}(N)$ is the market value \mathcal{M}_{-i} . Also by Assumption (A4), we have $R(\pi^{-0}(0) \cup \pi^{-0}(i)) \geq R(\pi^{-0}(0)) + R(\pi^{-0}(i))$. Therefore, it follows from $q_i = q_i^* = u^i(\pi^{-0}(i)) - V(N) + V_{-i}(N)$ that

$$\begin{aligned}
q_i &= u^i(\pi^{-0}(i)) - (R(\pi^{-0}(0)) + \sum_{l \in M} u^l(\pi^{-0}(l))) + V_{-i}(N) \\
&= R(\pi^{-0}(i)) + V_{-i}(N) - (R(\pi^{-0}(0)) + R(\pi^{-0}(i)) + \sum_{l \in M_{-i}} u^l(\pi^{-0}(l))) \\
&\geq R(\pi^{-0}(i)) + V_{-i}(N) - (R(\pi^{-0}(0) \cup \pi^{-0}(i)) + \sum_{l \in M_{-i}} u^l(\pi^{-0}(l))) \\
&\geq R(\pi^*(i)) + V_{-i}(N) - V_{-i}(N) \\
&= R(\pi^{-0}(i)).
\end{aligned}$$

Finally we prove (iii). Suppose that bidder i uses an arbitrary strategy $\bar{\sigma}_i$ rather than his sincere bidding strategy σ_i , and his opponents use their sincere bidding strategies $\{\sigma_l\}_{l \in M_{-i}}$. If this strategical behavior of bidder i results in the auction terminating in Step 4, it is obvious that

$$W_i(\sigma_i, \{\sigma_l\}_{l \in M_{-i}}, \{u^l\}_{l \in M}) \geq 0 \geq W_i(\bar{\sigma}_i, \{\sigma_l\}_{l \in M_{-i}}, \{u^l\}_{l \in M}),$$

for every realization $\{u^l\}_{l \in M}$. Therefore, it is sufficient to consider the case that the auction stops in Step 3. Assume that the auction terminates at round $t = \bar{t}$, and the auctioneer finds an allocation $\bar{\pi}^{-j}$ in every market \mathcal{M}_{-j} . By Theorem 4.2, we see that when every bidder $l \in M_{-i}$ bids sincerely, the pair $(\bar{p}^{-i}(\bar{t}), \bar{\pi}^{-i})$ is a Walrasian equilibrium for the market \mathcal{M}_{-i} . And so the allocation $\bar{\pi}^{-i}$ is efficient for \mathcal{M}_{-i} . Recall from the discussion of the above proof (i) that for every realization $\{u^l\}_{l \in M}$, we have

$$R(\bar{\pi}^{-i}(0)) + \sum_{l \in M_{-i}} u^l(\bar{\pi}^{-i}(l)) = R(\pi^{-i}(0)) + \sum_{l \in M_{-i}} u^l(\pi^{-i}(l))$$

because both allocations $\bar{\pi}^{-i}$ and π^{-i} are efficient for \mathcal{M}_{-i} , and

$$R(\bar{\pi}^{-0}(0)) + \sum_{l \in M} u^l(\bar{\pi}^{-0}(l)) \leq R(\pi^{-0}(0)) + \sum_{l \in M} u^l(\pi^{-0}(l))$$

because the allocation π^{-0} is efficient for \mathcal{M}_{-0} and $\bar{\pi}^{-0}$ may be not. Therefore, it holds that

$$\begin{aligned} & W_i(\bar{\sigma}_i, \{\sigma_l\}_{l \in M_{-i}}, \{u^l\}_{l \in M}) \\ &= u^i(\bar{\pi}^{-0}(i)) - R(\bar{\pi}^{-i}(0)) - \sum_{l \in M_{-i}} \bar{p}^{-i}(\bar{t}, \bar{\pi}^{-i}(l)) \\ &\quad + R(\bar{\pi}^{-0}(0)) + \sum_{l \in M_{-i}} \bar{p}^{-0}(\bar{t}, \bar{\pi}^{-0}(l)) \\ &= u^i(\bar{\pi}^{-0}(i)) - R(\bar{\pi}^{-i}(0)) - \sum_{l \in M_{-i}} \bar{p}^l(\bar{t}, \bar{\pi}^{-i}(l)) \\ &\quad + R(\bar{\pi}^{-0}(0)) + \sum_{l \in M_{-i}} \bar{p}^l(\bar{t}, \bar{\pi}^{-0}(l)) \\ &= u^i(\bar{\pi}^{-0}(i)) - R(\bar{\pi}^{-i}(0)) - \sum_{l \in M_{-i}} (u^l(\bar{\pi}^{-i}(l)) - V^l(\bar{p}^l(\bar{t}))) \\ &\quad + R(\bar{\pi}^{-0}(0)) + \sum_{l \in M_{-i}} (u^l(\bar{\pi}^{-0}(l)) - V^l(\bar{p}^l(\bar{t}))) \\ &= R(\bar{\pi}^{-0}(0)) + \sum_{l \in M} u^l(\bar{\pi}^{-0}(l)) - R(\bar{\pi}^{-i}(0)) - \sum_{l \in M_{-i}} u^l(\bar{\pi}^{-i}(l)) \\ &= R(\bar{\pi}^{-0}(0)) + \sum_{l \in M} u^l(\bar{\pi}^{-0}(l)) - R(\pi^{-i}(0)) - \sum_{l \in M_{-i}} u^l(\pi^{-i}(l)) \\ &\leq R(\pi^{-0}(0)) + \sum_{l \in M} u^l(\pi^{-0}(l)) - R(\pi^{-i}(0)) - \sum_{l \in M_{-i}} u^l(\pi^{-i}(l)) \\ &= V(N) - V_{-i}(N) \\ &= W_i(\sigma_i, \{\sigma_j\}_{j \in M_{-i}}, \{u^l\}_{l \in M}). \end{aligned}$$

This shows that sincere bidding is indeed an ex post Nash equilibrium in the auction game. \square

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